LECTURE 12: POLARITY CONTINUED; INTRODUCTION TO HAUSDORFF DISTANCE

We finish the proof of the following theorem. For completeness, we include here the first part of the proof.

Theorem 1. *Let $K \subset \mathbb{R}^n$ be a compact, convex set containing o in its interior. To any proper face F of K assign the set*

$$F^\circ = \{y \in K^* : \langle x, y\rangle = 1 \text{ for every } x \in F\}.$$

Then F° is a proper face of K^*, and the map $F \mapsto F^\circ$ is a bijection between the proper faces of K and K^* that reverses containment relation.

Proof. Let $H = \{y \in \mathbb{R}^n : \langle v_0, y\rangle = 1\}$ be an arbitrary supporting hyperplane of K satisfying $F = H \cap K$. Since $\langle v_0, y\rangle \leq 1$ for every $y \in K$ and $\langle v_0, y\rangle = 1$ for every $y \in F$, we have $v_0 \in F^\circ$. Thus, $F^\circ \neq \emptyset$. Now, let $x_0 \in \text{relint}(F)$ and $H' = \{y \in \mathbb{R}^n : \langle y, x_0\rangle = 1\}$. By the definition of polar set and $v_0 \in H'$, we have that H' is a supporting hyperplane of K^*, implying that $F' = K^* \cap H'$ is a proper face of K°. We show that $F' = F^\circ$.

By the definition of F°, $F^\circ \subset H'$ holds, and thus, $F^\circ \subseteq F'$. Now, let $y_0 \in K^* \setminus F^\circ$. Then, there is some $z \in F$ such that $\langle z, y_0\rangle < 1$. As $x_0 \in \text{relint}(F)$, there is a segment $[z, w] \subseteq F$ with $x_0 \neq w$. Then x_0 can be written in the form $x_0 = tz + (1-t)w$ for some $t \in (0, 1]$. But $w \in F$ and $y_0 \in K^*$ imply $\langle w, y_0\rangle \leq 1$, from which

$$\langle x_0, y_0\rangle t\langle z, y_0\rangle + (1-t)\langle w, y_0\rangle < 1,$$

that is, $y_0 \notin F'$. Thus, we have shown that $F^\circ = F'$ yielding, in particular, that $F \mapsto F^\circ$ is a face of K^*.

Now we prove that for any proper face F, we have $(F^\circ)^\circ = F$, which will imply that the map $F \mapsto F^\circ$ is injective. But since $(K^*)^* = K$; that is, applying this property for K^* we obtain that the map is bijective. By definition,

$$(F^\circ)^\circ = \{y \in (K^*)^* = K : \langle x, y\rangle = 1 \text{ for every } x \in F^\circ\}.$$

Thus, $F \subseteq (F^\circ)^\circ$. Let us consider the supporting hyperplane $H = \{y \in \mathbb{R}^n : \langle v_0, y\rangle = 1\}$ mentioned in the beginning of the proof. For this hyperplane $H \cap K = F$ is satisfied. During the proof we have shown that $v_0 \in F^\circ$. Hence, if $y \in (F^\circ)^\circ$, then $\langle y, v_0\rangle = 1$, but by the condition $H \cap K = F$ we have $y \in F$; that is, $(F^\circ)^\circ \subseteq F$.

We have shown that the map $F \mapsto F^\circ$ reverses the containment relation. But this property is a straightforward consequence of the definition of F°. $\square$

Definition 1. *Let $P, Q \subset \mathbb{R}^n$ be n-dimensional convex polytopes. We say that Q is a* dual *of P, if there is a bijection between the proper faces of Q and P that reverses containment.*

Problem 1. *Find dual pairs of polytopes P, Q.*

We remark that extending the above map to $\emptyset$ and the polytope itself, the duality of P and Q corresponds to the fact that the face lattices of P and Q are duals (cf. Definition 5 in the 8th lecture).

Proposition 1. *Let $P \subseteq \mathbb{R}^n$ be an arbitrary convex polytope. Then P has a dual polytope.*

Proof. Since translation and the dimension of the ambient space do not influence the existence of a dual polytope, we may assume that P is n-dimensional, and it contains o in its interior. But then P^* is a dual of P. $\square$

The following statement, which we present without proof, is often used in convex geometry. Before reading it, it is worth recalling that every compact set is Lebesgue measurable, and hence, it has a volume.

Proposition 2. *Let K be a compact, convex set containing o in its interior, and let $L : \mathbb{R}^n \to \mathbb{R}^n$ be a nondegenerate linear transformation. Then the quantity $V(L(K))V(L(K)^*)$ is independent of the choice of L, where the symbol $V(\cdot)$ denotes n-dimensional volume.*

Definition 2. *If $K \subseteq \mathbb{R}^n$ is a compact, convex set containing o in its interior, then the quantity $V(K)V(K^*)$ is called the* volume product *or* Mahler volume *of K.*

Theorem 2 (Blascke-Santaló)**.** *For any compact, convex set K with $K = -K$ and $o \in \operatorname{int} K$, we have*

$$V(K)V(K^*) \leq \kappa_n^2,$$

where κ_n denotes the volume of the n-dimensional unit ball.

The next conjecture is one of the most fundamental conjecture in convex geometry.

Conjecture 1 (Mahler)**.** *For any compact, convex set K with $K = -K$ and $o \in \operatorname{int} K$, we have*

$$V(K)V(K^*) \geq V(C)V(C^*),$$

where C is a cube centered at o.

Our next topic is Hausdorff distance. Let us recall the concepts of Minkowski sum and support function.

If $A, B \subseteq \mathbb{R}^n$ are nonempty sets, then their Minkowski sum is $A+B = \{a+b : a \in A, b \in B\}$. We have seen that if A, B are compact, convex sets, then $A+B$ is also compact and convex. We have defined the support function of a bounded set K as $h : \mathbb{R}^n \to \mathbb{R}$, $h_K(x) = \sup\{\langle x, y\rangle : y \in K\}$, and we have shown that if $o \in K$, then h_K is convex.

In the lecture we denote the family of compact, convex nonempty sets in $\mathbb{R}^n$ by $\mathcal{K}_n$. The main definition discussed in the lecture is the following.

Definition 3. *Let $K, L \in \mathcal{K}_n$ be compact sets. Then the* Hausdorff distance *of K and L is*

$$d_H(K, L) = \inf\{r \geq 0 : K \subseteq L + B_r(o) \text{ és } L \subseteq K + B_r(o)\}.$$

We remark that the above definition can be extended for bounded sets in general.

Proposition 3. *For any $K, L \in \mathcal{K}_n$, we have $d_H(K, L) = \sup\{|h_K(x) - h_L(x)| : x \in \mathbb{R}^n, ||x|| = 1\}$.*

Proof. By Proposition 2 in the 4th lecture, $h_{K+L} = h_K + h_L$. On the other hand, it can also be shown that for the above sets $K \subseteq L$ is satisfied if and only if $h_K(x) \leq h_L(x)$ is satisfied for all $x \in \mathbb{R}^n$. Indeed, by definition, $K \subseteq L$ implies the inequality for their support functions. Now, assume that $K \not\subseteq L$. Then there is a point $x \in K \setminus L$. Since L is compact and convex, x and L can be strictly separated by a hyperplane; that is, there is a unit vector $u \in \mathbb{R}^n$ and real number $\alpha \in \mathbb{R}$ such that $\langle x, u\rangle > \alpha$, and $\langle y, u\rangle < \alpha$ for any $y \in L$. But then $\sup\{\langle y, u\rangle : y \in K\} > \sup\{\langle y, u\rangle : y \in L\}$, implying $h_K(u) > h_L(u)$. Now, the statement readily follows by rephrasing the containment relations in the definition of Hausdorff distance. □

Proposition 4. *If $K, L, M \in \mathcal{K}_n$, then*

- *$d_H(K, L) \geq 0$, with equality if and only if $K = L$.*
- *$d_H(K, L) = d_H(L, K)$.*
- *$d_H(K, L) + d_H(L, M) \geq d_H(K, M)$.*

Proof. The inequality $d_H(K, L) \geq 0$ and the equality $d_H(K, K) = 0$ follows from the definition. On the other hand, if $d_H(K, L) = 0$, then $K \subseteq L$ and $L \subseteq K$, implying $K = L$. The definition does not disinguish the order of K and L, and thus, $d_H(K, L) = d_H(L, K)$. Finally, if $K \subseteq L + B_{r_1}(o)$ and $L \subseteq M + B_{r_2}(o)$, then $B_{r_1}(o) + B_{r_2}(o) = B_{r_1+r_2}(o)$

yields $K \subseteq M + B_{r_1+r_2}(o)$, and $M \subseteq L + B_{r_2}(o)$ and $L \subseteq K + B_{r_1}(o)$ implies similarly that $M \subseteq K + B_{r_1+r_2}(o)$. From this we obtain the triangle inequality $d_H(K, L) + d_H(L, M) \geq d_H(K, M)$. □

Corollary 1. *The family $\mathcal{K}_n$, equipped with Hausdorff distance, is a metric space.*

Let us recall that a metric space is called a *complete metric space* if every Cauchy sequence in the space is convergent. This property is investigated in the next theorem.

Theorem 3. *The family $\mathcal{K}_n$, equipped with Hausdorff distance, is a complete metric space.*

Proof. Let $K_i \in \mathcal{K}_n$, $i = 1, 2, \ldots$ be a Cauchy sequence of nonempty, compact, convex sets; i.e. assume that for every $\varepsilon > 0$ there is some $m_0 \in \mathbb{Z}^+$ such that if $m_1, m_2 > m_0$, then $d_H(K_{m_1}, K_{m_2}) < \varepsilon$. We show that then there is some $K \in \mathcal{K}_n$ such that $K_m \to K$ with respect to the topology induced by Hausdorff distance.

For every positive integer i, let $B_i = \mathrm{cl}(K_i \cup K_{i+1} \cup \ldots)$. By the properties of Cauchy sequences, B_i is a nonempty, bounded and closed set in $\mathbb{R}^n$, implying that it is compact, and $B_{i+1} \subseteq B_i$ for every i. Let $B = \cap_{i=1}^{\infty} B_i$. Since the intersection of arbitrarily many closed sets is closed, B is compact. We show that B is not empty. Indeed, if $B = \emptyset$, then the complements of the sets B_i with respect to the compact set B_1 form an open cover of B_1. But then we can choose a finite open subcover of B_1, i.e. there are finitely many B_is whose intersection is $\emptyset$, from which, as the sets are nested, it follows that $B_i = \emptyset$ for some value of i, which contradicts the definition of B_i. We have obtained that B is a nonempty, compact set.

Let $\varepsilon > 0$ be arbitrary. We show that there is an index $m \in \mathbb{Z}^+$ such that for every $i > m$, we have $B_i \subseteq \mathrm{int}(B + B_\varepsilon(o))$. By contradiction, suppose that it is not true. Then there is a sequence i_j of indices such that for every value of j, $B_{i_j} \not\subseteq \mathrm{int}(B + B_\varepsilon(o))$. Let $C_{i_j} = B_{i_j} \setminus \mathrm{int}(B + B_\varepsilon(o))$. By our conditions, the sets C_{i_j} are nonempty, nested, compact sets, which implies, as in the previous paragraph, $C = \cap_{i=1}^{\infty} C_{i_j}$ is a nonempty, compact set. But as the sets B_i are nested, $C \subseteq B_{i_j}$ for every value of j, implying that $C \subseteq B_i$ for every value of i. On the other hand, by their constructions, C and B are disjoint, which is a contradiction. Thus, for a suitable $m \in \mathbb{Z}^+$, $B_i \subseteq \mathrm{int}(B + B_\varepsilon(o))$ for all $i > m$. But from this it follows that $K_i \subseteq B + B_\varepsilon(o)$ for all $i > m$.

Since $\{K_i\}$ is a Cauchy sequence, there is an index k such that $d_H(K_i, K_j) < \varepsilon$ if $i, j > k$. Thus, if $i > k$ is arbitrary, then $\bigcup_{j=i}^{\infty} K_i \subseteq K_i + B_\varepsilon(o)$, implying $B \subseteq B_i \subseteq K_i + B_\varepsilon(o)$. This yields that if

$i > \max\{k, m\}$, then $d_H(B, K_i) \leq \varepsilon$, and thus, the limit set of $\{K_i\}$ is B.

We need to show that B is convex. Let $p, q \in B$ be arbitrary, and assume that for some $t \in (0, 1)$, $x = tp + (1 - t)q \notin B$. Then, by the compactness of B, there is a value $\delta > 0$ such that $B_\delta(x) \cap B = \emptyset$. Since the limit set of $\{K_i\}$ is B, there is an index i such that $K_i \subseteq B + B_{\delta/2}(o)$ and some points $p', q' \in K_i$ such that $||p - p'||, ||q - q'|| \leq \frac{\delta}{2}$. Let $x' = tp' + (1 - t)q' \in K_i$, which, by the triangle inequality, implies that $||x - x'|| \leq t||p - p'|| + (1 - t)||q - q'|| \leq \frac{\delta}{2}$, and thus, $x \in B_{\delta/2}(x')$. But from this we obtain $x \in K_i + B_{\delta/2}(o) \subseteq B + B_\delta(o)$, or in other words, $B_\delta(x) \cap B \neq \emptyset$, which is in contradiction with the choice of δ. □

Definition 4. *Let $\mathcal{F}$ be a nonempty family of nonempty sets in $\mathbb{R}^n$. If there is some $r > 0$ such that $F \subseteq B_r(o)$ for every $F \in \mathcal{F}$, then we say that $\mathcal{F}$ is* uniformly bounded.

The next theorem is a generalization of the Bolzano-Weierstrass theorem for bounded sequences.

Theorem 4 (Blaschke's Selection Theorem)**.** *Let $\mathcal{F} \subseteq \mathcal{K}_n$ be a uniformly bounded, infinite family. Then $\mathcal{F}$ contains a sequence converging to an element of $\mathcal{K}_n$.*

Proof. We show that $\mathcal{F}$ contains a Cauchy sequence. Let C be a cube in $\mathbb{R}^n$ that contains all elements of $\mathcal{F}$, and let the edge length of C be r. Let i be a positive integer, and dissect C with hyperplanes parallel to its facets into smaller (closed) cubes of edge length $\frac{r}{2^i}$. To any element K of $\mathcal{F}$, assign the union of the small cubes that intersect K. We call this set the ith minimal cover.

Since there are only finitely many possible first minimal covers, there is a union F_1 of small cubes which is the first minimal cover of infinitely many elements of $\mathcal{F}$. Legyen $\mathcal{F}_1 \subset \mathcal{F}$ be the subset of $\mathcal{F}$ whose first minimal cover is F_1. As $|\mathcal{F}_1| = \infty$ and there are only finitely many possible second minimal covers, there is a union F_2 of small cubes that is the second minimal cover of infinitely many elements of $\mathcal{F}_1$. Continuing this process, we obtained a sequence of nested subfamilies $\mathcal{F} \supseteq \mathcal{F}_1 \supseteq \mathcal{F}_2 \supseteq \ldots \supseteq \mathcal{F}_i \supseteq \ldots$ whith the property that every element of $\mathcal{F}_i$ has the same ith minimal cover F_i.

Let $K_i \in \mathcal{F}_i$, and consider the sequence $\{K_i\}$. According to the construction, for any $K_i \in \mathcal{F}_i$, $K_j \in \mathcal{F}_j$, $i < j$, the ith minimal cover of K_i and K_j i coincides. Since the diameters of the cubes forming an ith minimal cover is $\frac{r\sqrt{n}}{2^i}$, therefore then $d_H(K_i, K_j) \leq \frac{r\sqrt{n}}{2^i}$. But this implies that $\{K_i\}$ is a Cauchy sequence, and thus, by the previous theorem, it is convergent. □

According to the next theorem, the family of convex polytopes is an everywhere dense subfamily in $\mathcal{K}_n$.

Theorem 5. *Let $K \in \mathcal{K}_n$ be arbitrary. Then there is a sequence of convex polytopes $\{P_k\}$ that converges to K with respect to Hausdorff distance.*

Proof. Without loss of generality, assume that $\dim(K) = n$. To prove the statement, it is sufficient to show that for every $\varepsilon > 0$ there is some convex polytope P satisfying $P \subseteq K \subseteq P + B_\varepsilon(o)$, since choosing a polytope P_k for every positive integer k with the property that $P_k \subseteq K \subseteq P_k + B_{1/k}(o)$, the sequence $\{P_k\}$ satisfies the required conditions.

Since K is compact, there are points $x_1, \ldots, x_m \in K$ such that the open balls $\operatorname{int} B_\varepsilon(x_i)$ cover K. Let $P = \operatorname{conv}\{x_1, \ldots, x_m\}$. Then, clearly $P \subseteq K$. But $K \subseteq \bigcup_{i=1}^m \operatorname{int}(B_\varepsilon)(x_i) = \{x_1, \ldots, x_m\} + \operatorname{int} B_\varepsilon(o) \subseteq P + B_\varepsilon(o)$, from which the assertion follows. $\square$