LECTURE 5: MINKOWSKI SUM

Let us recall the definition of the Minkowski sum of two sets from the first lecture.

First, we prove the theorem stated in the last lecture.

Theorem 1. *Let $A \subset \mathbb{R}^n$ be an arbitrary bounded set containing o. Then the support function h_A of A is:*

(i) *convex, that is, $h(tx + (1-t)y) \leq th(x) + (1-t)h(y)$ for every $x, y \in \mathbb{R}^n$ and $t \in [0,1]$;*

(ii) *h nonnegative, and for any $\lambda \geq 0$ and $x \in \mathbb{R}^n$, we have $h(\lambda x) = \lambda h(x)$.*

Furthermore, for any function h satisfying the above properties there is a unique compact, convex set $A \subset \mathbb{R}^n$, containing o, whose support function is h.

Proof. Cearly,

$$h_A(tx + (1-t)y) = \sup\{\langle tx + (1-t)y, z\rangle : z \in A\} \leq$$

$$\leq t \sup\{\langle x, z\rangle : z \in A\} + (1-t)\sup\{\langle y, z\rangle : z \in A\} = th_A(x) + (1-t)h_A(y),$$

that is, h_A is convex. The second property readily follows from the properties of inner product.

Now, let h be a function satisfying (i) and (ii), and let $A = \{y \in \mathbb{R}^n : \langle x, y\rangle \leq h(x) \text{ for every } x \in \mathbb{R}^n\}$. As for any fixed x, the set of points y satisfying the inequality $\langle x, y\rangle \leq h(x)$ is a closed half space containing o, the set A, which is the intersection of such sets, is a closed, convex set containing o. We show that A is bounded, which will imply that it is compact. Suppose for contradiction that A is not bounded. Then there is some sequence $p_m \in A$, $p_m \neq o$, for which $||p_m|| \to \infty$. Since the boundary of a unit ball is compact, we can assume that there is some unit vector q satisfying $\frac{p_m}{||p_m||} \to q$. But the convexity and closedness of A yields that in this case $\frac{p_m}{||p_m||}, q \in A$ from which one can see that the half line $\{\lambda q : \lambda \in [0, \infty)\}$, starting at o and passing through q belongs to A. But then with the choice $x = q$ we have $\langle \lambda q, q\rangle \leq h(q)$ for any $\lambda \geq 0$, which is a contradiction. Thus, we have seen that A is compact. On the other hand, for any vector $z \in \mathbb{R}^n$, we have $h_A(z) = \sup\{\langle z, y\rangle : y \in A\} \leq h(z)$ by the definition of A.

We will show that $h_A(z) \geq h(z)$, that is, that there is a point $y \in A$, for which $\langle y, z\rangle = h(z)$. Since this statement clearly holds if $z = o$ or $h(z) = 0$, we assume that $z \neq o$ and $h(z) > 0$. Let us define the *epigraph* of h as the closed set $E_h = \{(x, \alpha) : h(x) \leq \alpha\} \subseteq \mathbb{R}^n \times \mathbb{R}$ (note that this set is the region 'above' the graph of h in $\mathbb{R}^{n+1}$). If $(x, \alpha), (y, \beta) \in E_h$ and $t \in [0, 1]$, then $h(tx + (1-)y) \leq th(x) + (1-t)h(y) \leq t\alpha + (1-t)\beta$, implying that E_h is convex, and clearly, if $(x, \alpha) \in E_h$ and $\lambda \geq 0$, then $(\lambda x, \lambda\alpha) \in E_h$. By the definition of epigraph, $(z, h(z))$ is a boundary pooint of E_h, and hence, by Corollary 4 of the first lecture, there are $(y, \beta) \in \mathbb{R}^n \times \mathbb{R}$ and $\alpha \in \mathbb{R}$ which satisfy $\langle y, w\rangle + \beta\gamma \leq \alpha$ for any $(w, \gamma) \in E_h$, and $\langle y, z\rangle + \beta h(z) = \alpha$. Since $z \neq o$, from the positive homogeneity of E_h it follows that $\alpha = 0$. On the other hand, since h is defined on the whole space $\mathbb{R}^n$, we have $\beta \neq 0$, and thus, with a suitable choice of y we may assume that $\beta = -1$. But from this $\langle y, z\rangle = h(z)$, which is what we wanted to prove. Thus, $h_A = h$.

Finally, we show that the support functions of different compact, convex sets containing o are different. Let A_1, A_2 be such sets. As $A_1 \neq A_2$, by suitably choosing indices there is a point $p \in A_1 \setminus A_2$. But then by Corollary 4 of the first lecture there is some $u \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$ such that $\langle u, p\rangle > \alpha$, and $\langle u, x\rangle \leq \alpha$ for every $x \in A_2$. But from this $h_{A_1}(u) > h_{A_2}(u)$ follows. □

Proposition 1. *For any convex sets $K, L \subset \mathbb{R}^n$, we have $h_{K+L} = h_K + h_L$.*

Proof. If $x \in \mathbb{R}^n$, then

$$h_{K+L}(x) = \sup\{\langle x, y\rangle + \langle x, z\rangle : y \in K, z \in L\} =$$
$$= \sup\{\langle x, y\rangle : y \in\} + \sup\{\langle x, z\rangle : z \in L\} = h_K(x) + h_L(x).$$

□