LECTURE 8: VALUATIONS AND THE EULER CHARACTERISTIC

Let us recall the following concept from our previous studies.

Definition 1. *Let $A \subset \mathbb{R}^n$ be a set. The* indicator function $I[A]$ *of the set is the function*

$$I[A](x) = \begin{cases} 1, & \text{if } x \in A, \\ 0, & \text{if } x \notin A. \end{cases}$$

We remark that for any $A, B \subset \mathbb{R}^n$, we have $I[A] \cdot I[B] = I[A \cap B]$.

Lemma 1 (Inclusion-exclusion formula). *For any sets $A_1, A_2, \ldots, A_k \subset \mathbb{R}^n$,*

$$I[A_1 \cup A_2 \cup \ldots \cup A_k] = 1 - (1 - I[A_1])(1 - I[A_2]) \ldots (1 - I[A_k]) =$$

$$= \sum_{j=1}^{k} (-1)^{j-1} \sum_{1 \le i_1 < i_2 < \ldots < i_j \le k} I[A_{i_1} \cap A_{i_2} \cap \ldots \cap A_{i_j}].$$

Proof. Let us introduce the notation $\bar{B} = \mathbb{R}^n \setminus B$ for any set $B \subseteq \mathbb{R}^n$. Observe that the first statement is equivalent to the equality

$$A_1 \cup A_2 \cup \ldots \cup A_k = \overline{\bar{A}_1 \cap \bar{A}_2 \cap \ldots \bar{A}_k},$$

which readily follows from the de Morgan identities. The second statement is a consequence of the previous remark. □

Definition 2. *The real vector space generated by the indicator functions$I[A]$ of the compact, convex sets $A \subset \mathbb{R}^n$ is called the* algebra of compact, convex sets, *and is denoted by $\mathcal{K}(\mathbb{R}^n)$. The real vector space generated by the indicator functions$I[A]$ of the closed, convex sets $A \subset \mathbb{R}^n$ is called the* algebra of closed, convex sets, *and is denoted by $\mathcal{C}(\mathbb{R}^n)$.*

Remark 1. *An arbitrary element of $\mathcal{K}(\mathbb{R}^n)$ can be written as $\sum_{i=1}^{k} \alpha_i I[A_i]$, where $\alpha_i \in \mathbb{R}$, and the sets $A_i \subset \mathbb{R}^n$ are compact and convex. Observe that if $A, B \subset \mathbb{R}^n$ are compact, convex sets, then $A \cap B$ is also compact and convex, implying that the product of two elements of $\mathcal{K}(\mathbb{R}^n)$ is also an element of $\mathcal{K}(\mathbb{R}^n)$. Thus, the set $\mathcal{K}(\mathbb{R}^n)$ is indeed an algebra over $\mathbb{R}$. A similar observation can be made about the algebra $\mathcal{C}(\mathbb{R}^n)$.*

Definition 3. *A linear map $\mathcal{K}(\mathbb{R}^n) \to \mathbb{R}$ or $\mathcal{C}(\mathbb{R}^n) \to \mathbb{R}$ is called a* valuation.

The main goal of this lecture is the proof of the next theorem.

Theorem 1. *There is a unique valuation $\chi : \mathcal{C}(\mathbb{R}^n) \to \mathbb{R}$ satisfying $\chi(I[A]) = 1$ for all nonempty, closed, convex sets $A \subset \mathbb{R}^n$.*

This valuation is called the *Euler characteristic* induced by the algebra of closed, convex sets. Theorem 1 was first proved by H. Hadwiger.

Proof. Note that by the linearity of χ, it can be uniquely extended to every element of $\mathcal{C}(\mathbb{R}^n)$, implying that χ is unique. We need to show that χ exists. We first define this valuation on the elements of $\mathcal{K}(\mathbb{R}^n)$ by induction on the dimension.

Assume that $n = 0$. Then any function $f \in \mathcal{K}(\mathbb{R}^0)$ can be written as $f = \alpha I[o]$ for some $\alpha \in \mathbb{R}$. Thus, $\chi(f) = \alpha$ satisfies the conditions of the theorem.

Let $n > 0$. Fo any $x \in \mathbb{R}^n$, let $p(x)$ denote the last coordinate od x, and for any $t \in \mathbb{R}$, define the hyperplane

$$H_t = \{x \in \mathbb{R}^n : p(x) = t\}.$$

This hyperplane can be identified with $\mathbb{R}^{n-1}$, and thus, there is a (unique) valuation χ_t on it satisfying the conditions of the theorem. For any $f \in \mathcal{K}(\mathbb{R}^n)$, let f_t denote the restriction of f onto H_t. Then, if $f = \sum_{i=1}^k \alpha_i I[A_i]$, where $\alpha_i \in \mathbb{R}$ and the A_is are compact, convex sets, then

$$f_t = \sum_{i=1}^k \alpha_i I[A_i \cap H_t],$$

and hence, by $f_t \in \mathcal{K}(H_t)$, we have

$$\chi_t(f_t) = \sum_{i : A_i \cap H_t \neq \emptyset} \alpha_i.$$

Consider the limit

$$\lim_{\varepsilon \to 0^+} \chi_{t-\varepsilon}(f_{t-\varepsilon}).$$

Note that this limit is equal to $\chi_t(f_t)$ if and only if for any sufficiently small $\varepsilon > 0$ and for every value of i, $A_i \cap H_t \neq \emptyset$ implies $A_i \cap H_{t-\varepsilon} \neq \emptyset$.

In general, we have that $\lim_{\varepsilon \to 0^+} \chi_{t-\varepsilon}(f_{t-\varepsilon})$ is equal to the sum of the α_is for which, for any small $\varepsilon > 0$, we have $A_i \cap H_{t-\varepsilon} \neq \emptyset$. That is, the limit is $\chi_t(f_t)$ unless t is the minimum of the orthogonal projection p on a set A_i. Thus, for any function f, the limit differs from $\chi_t(f_t)$ only for finitely many values of t. Based on this, we define the function χ as

$$\chi(f) = \sum_{t \in \mathbb{R}} \left(\chi_t(f_t) - \lim_{\varepsilon \to 0^+} \chi_{t-\varepsilon}(f_{t-\varepsilon}) \right).$$

Consider the functions $f, g \in \mathcal{K}(\mathbb{R}^n)$ and numbers $\alpha, \beta \in \mathbb{R}$. Since the valuation χ_t, and the operation of taking limit, are linear, it follows that $\chi(\alpha f + \beta g) = \alpha\chi(f) + \beta\chi(g)$. Furthermore, if $A \subset \mathbb{R}^n$ is a nonempty, compact, convex set, then

$$\chi_t(I[A \cap H_t]) - \lim_{\varepsilon \to 0^+} \chi_{t-\varepsilon}(I[A \cap H_{t-\varepsilon}]) = \begin{cases} 1, & \text{if } \min_{x \in A} p(x) = t, \\ 0, & \text{otherwise.} \end{cases}$$

As the minimum is uniquely defined on A, we have $\chi(I[A]) = 1$.

Now we extend χ to $\mathcal{C}(\mathbb{R}^n)$. Using the standard notation $B_\rho(o) = \{x \in \mathbb{R}^n : ||x|| \leq \rho\}$, if $f \in \mathcal{C}(\mathbb{R}^n)$, let

$$\chi(f) = \lim_{\rho \to \infty} f \cdot I[B_\rho(o)].$$

Then χ clearly satisfies the requirements. □

If $A \subset \mathbb{R}^n$ is a set such that $I[A] \in \mathcal{C}(\mathbb{R}^n)$, then, instead of $\chi(I[A])$, we use the notation$\chi(A)$. We call this quantity the *Euler characteristic of A*. We remark that Euler characteristic can be also defined in a more general setting, for the so-called *CW complexes*. Nevertheless, the discussion of these complexes is outside the scope of this course.

In the proof of the previous theorem, we proved also the following lemma.

Lemma 2. *Let $A \subset \mathbb{R}^n$ be a set such that $I[A] \in \mathcal{K}(\mathbb{R}^n)$. Let $t \in \mathbb{R}$, and let H_t be the set of the points $x = (x_1, \ldots, x_n)$ with $x_n = t$. Then $I[A \cap H_t] \in \mathcal{K}(\mathbb{R}^n)$, and*

$$\chi(A) = \sum_{t \in \mathbb{R}} \left(\chi(A \cap H_t) - \lim_{\varepsilon \to 0^+} \chi(A \cap H_{t-\varepsilon}) \right).$$

The last lemma is the consequence of Lemma 1 of the sixth lecture, and Theorem 1.

Lemma 3. *Let $A_1, A_2, \ldots, A_k \subset \mathbb{R}^n$ be sets such that $I[A_i] \in \mathcal{K}(\mathbb{R}^n)$ for any $i = 1, 2, \ldots, k$. Then*

$$\chi(A_1 \cup A_2 \cup \ldots \cup A_k) = \sum_{j=1}^{k} (-1)^{j-1} \sum_{1 \leq i_1 < i_2 < \ldots < i_j \leq k} \chi(A_{i_1} \cap A_{i_2} \cap \ldots \cap A_{i_j}).$$