Convex Geometry tutorial
For students with mathematics major

Problem sheet 2 - Convex hull, theorems of Radon and Carathéodory - Solutions

Exercise 1. Prove that if $A \subseteq B$, then $\operatorname{conv} A \subseteq \operatorname{conv} B$.

Solution

If a convex set contains B, then it clearly contains A, and thus, the statement follows from the definition of convex hull.

Exercise 2. A set $S \subseteq \mathbb{R}^n$ is called a *convex cone* if it is convex and for every $x \in S$ the points λx, $\lambda \geq 0$ are elements of S. Following the definition of convex combination and convex hull, define the conic combination of points and the conic hull of a set. Show that a conic hull is a convex cone, and it coincides with the set of the conic combinations of the finite subsets of the set.

Solution

Let us define the conic hull of a set S as the intersection of all convex cones containing S. We show that this set is a convex cone. Since a convex cone is convex, it is sufficient to show that if x is contained in the intersection of all convex cones containing S, then the same holds for all λx, $\lambda \geq 0$. But by the definition of convex cones it is satisfied for all convex cones containing S, and hence it is satisfied for the intersection of these sets.

Let us define a conic combination of the points $p_1, \ldots, p_k \in \mathbb{R}^n$ as the points $\sum_{i=1}^k \alpha_i p_i$, where $\alpha_i \geq 0$ for all is. Let C denote the set of all conic combinations of the finite subsets of S, and let C' denote the conic hull of S. Since a convex combination of conic combinations is a conic combination, we have that C is convex. Furthermore, if $\lambda \geq 0$, then λ times a conic combination is a conic combination of the same points, implying that C is a convex cone. Thus, $C' \subseteq C$. On the other hand, observe that as C' is convex it contains the convex combinations of all finite subsets of S. But a conic combination of some points of S can be written as a convex combination of the same points, multiplied by some suitably chosen $\lambda \geq 0$. Thus, the fact that C' is a convex cone yields that C' contains the conic combinations of the finite subsets of S; that is, $C \subset C'$.

Exercise 3. A set $K \subset \mathbb{R}^n$ is called *locally convex* if for every $p \in K$ there is some $\rho > 0$ such that the intersection of K with the ball $B(p, \rho)$ of radius ρ and center p is convex. Is it true that every locally convex set is convex?

Solution

The answer is no. As examples for locally convex but not convex sets we can take any finite point set.

Exercise 4. Give an example for a closed set $A \subseteq \mathbb{R}^2$ whose convex hull is not closed.

Solution

Let $A = \{(x, 0) \in \mathbb{R}^2 : x \in \mathbb{R}\} \cup \{(0, 1)\}$. Then A is a closed set, but $\operatorname{conv}(A) = \{(x, y) \in \mathbb{R}^2 : x \in \mathbb{R}, 0 \leq y < 1\} \cup \{(0, 1)\}$, which is not closed.

Exercise 5. Prove that the convex hull of an open set is open.

Solution

Let A be open, and let $p \in \operatorname{conv}(A)$. Then there are some points $p_1, \ldots, p_k \in A$ and real numbers $\alpha_i \geq 0$, $\sum_{i=1}^k \alpha_i = 1$ such that $p = \sum_{i=1}^k \alpha_i p_i$. Since A is open, there is some $\rho > 0$ with the property that A contains the closed ball of radius ρ centered at p_i for all values of i. In other words, there is some $\rho > 0$ such that for every x with $||x|| \leq \rho$ and every value of i, we have $p_i + x \in A$. But then

$\sum_{i=1}^k \alpha_i(p_i+x) = \left(\sum_{i=1}^k \alpha_i p_i\right) + x = p + x$, from which we have $p+x \in \operatorname{conv}(A)$. Thus, $\operatorname{conv}(A)$ contains the closed ball of radius ρ and center p. Since $p \in \operatorname{conv}(A)$ was arbitrary, this implies that $\operatorname{conv}(A)$ is open.

Exercise 6. Let $S \subset \mathbb{R}^n$ be a set consisting of $n+2$ points in general position (i.e. any $n+1$ of the points is affinely independent). Prove that then S can be uniquely decomposed into two disjoint subsets S_1, S_2 satisfying $\operatorname{conv} S_1 \cap \operatorname{conv} S_2 \neq \emptyset$. In addition, prove that in this case the intersection is a singleton.

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Let $S = \{p_1, p_2, \ldots, p_{n+2}\}$, where every $(n+1)$-element subset of S is affinely independent.Then the homogeneous system of linear equations $\sum_{i=1}^{n+2} \alpha_i p_i = o$, $\sum_{i=1}^{n+2} \alpha_i = 0$ (containing $n+1$ equations and $n+2$ variables) has a nontrivial solution, which, by the Kronecker-Capelli theorem, can be given using exactly one free parameter; that is, there is some vector $(\beta_1, \beta_2, \ldots, \beta_{n+2})$ with not all β_is equal to zero such that the solutions consists of the vectors $\alpha_i = t\beta_i$, $t \in \mathbb{R}$. If there was some i, for which $\beta_i = 0$, then by Theorem 1 in the first lecture the remaining $n+1$ points would be affinely dependent, implying that no β_i is zero. Let $U = \{i : \beta_i > 0$, $V = \{i : \beta_i < 0\}$, and for every i let $\gamma_i = -\beta_i$. Then $\sum_{i\in U} \beta_i = \sum_{i \in V} \gamma_i$.

Now we prove the statement. Assume that the index set $\{1, 2 \ldots, n+2\}$ has a decomposition into disjoint sets U, V, and there are some coefficients $\alpha_i \geq 0$, $\delta_i \geq 0$, $\sum_{i\in U} \alpha_i = \sum_{i\in V} \beta_i = 1$ that satisfy the conditions $\sum_{i\in U} \alpha_i p_i = \sum_{i \in V} \delta_i p_i$. Then, introducing the notation $\alpha_i = -\delta_i$ for $i \in V$, the above point can be assigned to a solution of the system of the linear equations $\sum_{i=1}^{n+2} \alpha_i = 0$, $\sum_{i=1}^{n+2} \alpha_i p_i = o$. But according to the description of the solution in the previous paragraph, both the sets U, V and the coefficients assigned to such a decomposition, are determined uniquely.

Exercise 7. * Let $\sigma \in S_n$ be a permutation. Define the permutation matrix assigned to σ by $A_\sigma := (a_{ij})$, where

$$a_{ij} = \begin{cases} 1, & \text{if } \sigma(i) = j \\ 0, & \text{if } \sigma(i) \neq j. \end{cases}$$

A matrix $B = (b_{ij})$ is called *doubly stochastic*, if its entries are nonnegative, and the sum of the entries in each row and each column is one. Prove that the convex hull of the set of permutation matrices in $\mathbb{R}^{n^2}$ is the set of doubly stochastic matrices. (Hint: try to reduce the problem to Hall's theorem for bipartite graphs)

Solution

Let S denote the set of permutation matrices and C denote the set of doubly stochastic matrices in $\mathbb{R}^{n^2}$. We need to show that $\operatorname{conv}(S) = C$. First we prove that the C is convex. Let $A = (a_{ij})$ and $B = (c_{ij})$ be doubly stochastic, and consider the matrix $C = (c_{ij}) = tA + (1-t)B$ for some $t \in [0,1]$. Then, clearly, the elements of C are nonnegative. The sum of the elements of C in the ith row is

$$\sum_{j=1}^n c_{ij} = \sum_{j=1}^n (ta_{ij} + (1-t)b_{ij}) = t\sum_{j=1}^n a_{ij} + (1-t)\sum_{j=1}^n b_{ij} = t \cdot 1 + (1-t)\cdot 1 = 1,$$

implying that C is doubly stochastic. By the definition of convex sets, this yields that C is convex.

On the other hand, any permutation matrix is doubly stochastic, and thus, $S \subseteq C$, implying that $\operatorname{conv}(S) \subseteq C$. Thus, we need to show that every element of C is a convex combination of permutation matrices. Let $D = (d_{ij})$ be a doubly stochastic matrix different from any permutation matrix. We define a weighted graph $G = G(V,E)$ as follows. The vertices of G are $V = \{r_1, \ldots, r_n, c_1, \ldots, c_n\}$ (corresponding to the rows and columns of D), and the edges of G are the pairs $\{r_i, c_j\}$ with $d_{ij} \neq 0$, and in this case the weight $w(r_i, c_j)$ of $\{r_i, c_j\}$ is d_{ij}. Clearly, G is a bipartite graph. Let $R = \{r_1, \ldots, r_n\}$ and $C = \{c_1, \ldots, c_n\}$. For any $A \subseteq V$ let the *neighborhood* $N(A)$ of A be defined as the vertices of G connected to at least one vertex of A. Then we have $N(A) \subseteq C$ for any $A \subseteq R$ and $N(A) \subseteq R$ for any $A \subseteq C$. Furthermore, if $A \subseteq R$, then

$$\sum_{r_i \in A, c_j \in N(A)} w(r_i, c_j) = \sum_{r_i \in A} \sum_{c_j \in N(\{r_i\})} w(r_i, c_j) = \sum_{r_i \in A} \sum_{c_j \in N(\{r_i\})} d_{ij} = \sum_{r_i \in A} 1 = |A|,$$

and the same statement holds if $A \subseteq C$. On the other hand, for any $A \subseteq R$ or $A \subseteq C$, we have $A \subseteq N(N(A))$ by the definition of neighborhood. Thus, assuming that $A \subseteq R$,

$$|N(A)| = \sum_{c_j \in N(A), r_i \in N(N(A))} w(r_i, c_j) \geq \sum_{c_j \in N(A), r_i \in A} w(r_i, c_j) = |A|,$$

and the same inequality holds if $A \subseteq C$. Hence, we may apply Hall's theorem for bipartite graphs which states that in this case G has a perfect matching: there is a permutation σ of $\{1, \ldots, n\}$ such that $\{r_i, c_{\sigma(i)}\}$ is an edge of G. Let P be the $n \times n$ permutation matrix defined by σ. Note that by our construction, for every value of i, $d_{i\sigma(i)} > 0$, and since $D \neq P$, we have $d_{i\sigma(i)} < 1$ for some value of i. For any $t \in [0, 1)$ let $D(t)$ be the matrix defined by $D = (1-t)D(t) + tP$, or in other words, let $D(t) = \frac{1}{1-t}D - \frac{t}{1-t}P$. Observe that $D(0) = D$ and that $D(t)$ has a negative entry if t is sufficiently close to 1. Thus, there is a maximal value t_0 such that $D(t_0)$ has only nonnegative entries, which implies by the continuity of the entries of $D(t)$ that for all ij, $d_{ij} = 0$ implies that the corresponding entry of $D(t_0)$ is zero, and also that some other entry of $D(t_0)$ is also zero. On the other hand, $D(t_0)$ is a doubly stochastic matrix by our construction, and hence, D can be written as the convex combination of a permutation matrix and a doubly stochastic matrix with strictly more zero entries. By continuing this process, we can write D as a convex combination of permutation matrices.