Convex Geometry tutorial
for students with mathematics major

Problem sheet 6 - Algebra of convex sets, the Euler characteristic - Solutions

Exercise 1. Prove that if $f, g \in \mathcal{K}(\mathbb{R}^n)$, then $fg \in \mathcal{K}(\mathbb{R}^n)$, and if $f, g \in \mathcal{C}(\mathbb{R}^n)$, then $fg \in \mathcal{C}(\mathbb{R}^n)$.

Solution.
Let $f = \sum_{i=1}^{k} \alpha_i I[K_i]$ and $g = \sum_{j=1}^{m} \beta_j I[L_j]$, where all coefficients are real numbers, and the sets K_i and L_j are compact/closed convex sets. Then

$$fg = \left(\sum_{i=1}^{k} \alpha_i I[K_i]\right)\left(\sum_{j=1}^{m} \beta_j I[L_j]\right) = \sum_{i=1}^{k}\sum_{j=1}^{m} \alpha_i \beta_j I[K_i] I[L_j] =$$

$$= \sum_{i=1}^{k}\sum_{j=1}^{m} \alpha_i \beta_j I[K_i \cap L_j].$$

Since the intersecion of compact/closed convex sets is compact/closed and convex, if $f, g \in \mathcal{K}(\mathbb{R}^n)$, then $fg \in \mathcal{K}(\mathbb{R}^n)$, and if $f, g \in \mathcal{C}(\mathbb{R}^n)$, then $fg \in \mathcal{C}(\mathbb{R}^n)$.

Exercise 2. Is it true that the indicator functions of compact, convex sets form a basis of the vector space $\mathcal{K}(\mathbb{R}^n)$?

Solution.
By the definition of $\mathcal{K}(\mathbb{R}^n)$, the indicator functions of compact, convex sets form a generating system of the vector space $\mathcal{K}(\mathbb{R}^n)$. On the other hand, it is not true that indicator function are linearly independent. For this, it is sufficient to show that using suitable sets and coefficients, the constant zero function can be written as a nontrivial linear combination of indicator functions of nonempty sets.An example for this is the following:
Let $p, q, r \in \mathbb{R}^n$ be collinear points satisfying $q \in [p, r]$. Let $S_1 = [p, q]$, $S_2 = [q, r]$ and $S_3 = [p, r]$. But then $I[S_1] + I[S_2] - I[\{q\}] = I[S_3]$, or in other words, $I[S_1] + I[S_2] - I[\{q\}] - I[S_3] = 0$.

Exercise 3. Let $K_1, K_2, K_3 \subset \mathbb{R}^n$ be closed, convex sets whose union is convex. Prove that if the intersection of any pair of them is nonempty, then the intersection of all three sets is nonempty.

Solution.
The union of finitely many closed sets is closed, and the intersection of closed, convex sets is closed and convex. Thus, by the Inclusion-Exclusion formula and the definition of Euler characteristic, we have

$$1 = \chi(K_1 \cup K_2 \cup K_3) = \chi(K_1) + \chi(K_2) + \chi(K_3) - \chi(K_1 \cap K_2) - \chi(K_1 \cap K_3) - \chi(K_2 \cap K_3) + \chi(K_1 \cap K_2 \cap K_3) =$$

$$= \chi(K_1 \cap K_2 \cap K_3).$$

Thus, $K_1 \cap K_2 \cap K_3 \neq \emptyset$, as otherwise $\chi(K_1 \cap K_2 \cap K_3) = 0$.

Exercise 4. Let $K_1, K_2, \dots K_m \subset \mathbb{R}^n$ be closed, convex sets whose union is convex. Prove that if any k of them have a nonempty intersection, then there are $k+1$ sets among them whose intersection is not empty.

Solution.
The conditions can be valid only if we assume that $1 \leq k \leq m-1$, and $m \geq 2$.

We use the idea of the previous exercise. Observe that the set $\bigcup_{i=1}^m K_i$ is closed and convex, and assume that for some value $1 \leq k \leq m-1$, any k of these sets have a nonempty intersection. Then, following te consideration in the previous exercise, we obtain the equation $1 = m - \binom{m}{2} + \ldots + (-1)^{k-1}\binom{m}{k}$ from which $\sum_{j=0}^k (-1)^j \binom{m}{j} = 0$. It is a well-known identity for binomial coefficients that $\binom{a}{b} = \binom{a-1}{b-1} + \binom{a-1}{b}$ for any $a \geq 2$ and $1 \leq b \leq a-1$. Thus,

$$\sum_{j=0}^k (-1)^j \binom{m}{j} = \binom{m-1}{0} + \sum_{j=1}^k (-1)^j \left(\binom{m-1}{j-1} + \binom{m-1}{j} \right) = (-1)^k \binom{m-1}{k},$$

which cannot be equal to zero.

Exercise 5. Define the volume $V(\cdot)$ of a set in the usual way, i.e. as its Lebesgue measure. Using the fact that (being Borel sets), any compact, convex set is Lebesgue measurable, prove that for any $\alpha_1, \alpha_2, \ldots, \alpha_m \in \mathbb{R}$ and compact, convex sets $K_1, K_2, \ldots, K_m \in \mathbb{R}^n$, if $\sum_{i=1}^m \alpha_i I[K_i] = 0$, then $\sum_{i=1}^m \alpha_i V(K_i) = 0$.

Solution.
For arbitrary $S = (i_1, \ldots, i_m) \in \{0,1\}^m$, define the set K_S as $K_S = \left(\bigcap_{i_j=1} K_{i_j}\right) \setminus \left(\bigcup_{i_j=0} K_{i_j}\right)$. This set is the set of points contained in exactly those of the K_{i_j}s where i_j (that is, the jth coordinate of S) is 1.Observe that these sets are disjoint and measurable. Furthermore, let $\beta_S = \sum_{i_j=1} \alpha_{i_j}$. Then $\sum_{i=1}^m \alpha_i V(K_i) = \sum_{S \in \{0,1\}^m} \beta_S V(K_S)$. Note that by the conditions, if $K_S \neq \emptyset$, then $\beta_S = 0$. Thus, every member of the above sum is zero, implying that $\sum_{i=1}^m \alpha_i V(K_i) = 0$.

Exercise 6. Prove that there is a valuation on $\mathcal{K}(\mathbb{R}^n)$ whose value at the indicator function of any compact, convex set K is $V(K)$.

Solution.
If $f \in \mathcal{K}(\mathbb{R}^n)$, then f can be written as $f = \sum_{i=1}^m \alpha_i I[K_i]$, where $K_1, \ldots, K_m \subset \mathbb{R}^n$ are compact, convex sets, and $\alpha_1, \ldots, \alpha_m \in \mathbb{R}$. Define the valuation $\varphi : \mathcal{K}(\mathbb{R}^n) \to \mathbb{R}$ as $\varphi(f) = \sum_{i=1}^m \alpha_i V(K_i)$. Since the indicator functions of compact, convex sets generate the algebra $\mathcal{K}(\mathbb{R}^n)$, in this way we have defined the valuation at every element of $\mathcal{K}(\mathbb{R}^n)$. Thus, we need to show that the above definition is contradiction-free, that is, if the same function $f \in \mathcal{K}(\mathbb{R}^n)$ is represented in two different ways, then we assign the same value to it. Rearranging the equality of these representations to obtain the constant zero function on one side, this means that we need to prove that if $\sum_{i=1}^m \alpha_i I[K_i] = 0$, then $\sum_{i=1}^m \alpha_i V(K_i) = 0$. But this follows from the solution of the previous exercise.