Convex Geometry tutorial
for students with mathematics major

Problem sheet 7 - Polytopes, polyhedral sets, their face structures - Solutions

Exercise 1. Prove that the extremal points of a polytope are also its exposed points.

Solution.
In Theorem 1 of the 8th lecture we have proved exactly this.

Exercise 2. A proper face of a closed, convex set is its intersection with one of its supporting hyperplanes. Prove that a proper face of a polyhedral set is also a polyhedral set.

Solution.
Let $P = \bigcap_{i=1}^m H_i$ be a polyhedral set, where the sets H_i are closed half spaces, and let F be a proper face of P. Then there is a supporting hyperplane H of P such that $H \cap P = F$. But if the two closed half spaces bounded by H are denoted by H^+ and H^-, then $F = P \cap H \left(\bigcap_{i=1}^m H_i\right) \cap H^+ \cap H^-$, therefore F is the intersection of finitely many closed half spaces.

Exercise 3. Let $K = \{p : \langle p, u_i \rangle \geq \alpha_i, i \in I\}$, with $|I| < \infty$, be a bounded polyhedral set. For any point $p \in K$ let $I(p) = \{i : \langle p, u_i \rangle = \alpha_i\}$. Let $F = \{q \in K : \langle q, u_i \rangle = \alpha_i : i \in I(p)\}$. Prove that if we regard K as a face of itself, then F is the smallest face, with respect to inclusion, such that $p \in F$.

Solution.
By the continuity of linear functionals and the finiteness of I, if $p \in \text{int}(K)$, then $I(p) = \emptyset$ and $F = K$, implying the statement. Assume that $p \in \text{bd}(K)$. Since every bounded polyhedral set is a convex polytope, (cf. Theorem 3 in the 8th lecture), by Lemma 2 of the 9th lecture (and its proof) there is a unique face of K whose relative interior contains p, and this face is the intersection of all the faces containing p. Thus, to prove the statement it is sufficient to prove that $p \in \text{relint}(F)$.

Let $X = \text{aff}(F) = \{q \in \mathbb{R}^n : \langle q, u_i \rangle = \alpha_i : i \in I(p)\}$. Then the restrictions of the linear functionals, defining K, to X are either linear functionals or constants. More specifically, if $i \in I(p)$, then the restriction of the ith linear functional is constant, and otherwise it is a linear functional. Thus, F is the set $F = \{q \in X : \langle q, u_i \rangle \geq \alpha_i : i \in I \setminus I(p)\}$. But by the definition of $I(p)$, $\langle q, u_i \rangle > \alpha_i$ for every $i \in I \setminus I(p)$, and hence, $q \in \text{relint}(F)$.

Exercise 4. Prove that every n-dimensional polytope has a facet. Prove that for every $k = 0, 1, \ldots, n-1$, every n-dimensional polytope has a k-dimensional face.

Solution.
By Corollary 2 of the 8th lecture, the boundary of an n-dimensional polytope is the union of its facets. Thus, P has a facet. Since the faces of a convex polytope are convex polytopes, and faces of a face of P are faces of P (Proposition 3 in the 8th lecture), the second statement can be proved by induction on the dimension.

Exercise 5. Prove that an $(n-2)$-dimensional face of an n-dimensional polytope belongs to exactly two facets.

Solution.
Let G be an $(n-2)$-dimensional face of the n-dimensional polytope $P \subset \mathbb{R}^n$, where, for simplicity, we assume that $o \in G$. Let N be the set of the outer normal vectors of the supporting hyperplanes containing G. Then N is a subset of the orthogonal complement $(\text{aff}(G))^\perp$ of $\text{aff}(G)$, which is a 2-dimensional plane. Thus, the orthogonal projection of P onto $(\text{aff}(G))^\perp$ is a convex polygon, and

G is one of its vertices. Furthermore, by $N \subseteq (\mathrm{aff}(G))^{\perp}$, the set of the outer normal vectors of the supporting lines of this convex polygon at this vertex is N. Thus, the supporting hyperplane perpendicular to a relative interior point of N intersects P in G, and the one perpendicular to a point in either of the two rays forming the relative boundary of N is a proper face of P, respectively, that strictly contains G. Thus, these two faces are facets containing G. On the other hand, the outer normal vectors of any facet containing G lie in N, which shows that there are no more facets of P containing G.

Exercise 6. (Diamond property) Let P be an arbitrary n-dimensional polytope, and $F \subset G$ are faces of P with $\dim F + 2 = \dim G$. Then P has exactly two faces F_1, F_2, different from F and G, that satisfy $F \subset F_1, F_2 \subset G$.

Solution.

Observe that the dimension of the faces satisfying the conditions is $\dim(F) + 1 = \dim(G) - 1$. Thus, the statement follows from applying the result of the previous exercise to the polytope G.