Convex Geometry tutorial
for students with mathematics major

Problem sheet 8 - Euler's theorem - Solutions

Exercise 1. Let $P \subset \mathbb{R}^n$ be an n-dimensional convex polytope. Let H be a hyperplane, passing through an interior point of P, which does not contain any vertex of P. Let H^+ be one of the two open half spaces bounded by H, and let f_i^+ denote the number of the i-dimensional faces of P contained in H^+. Then

$$\sum_{i=0}^{n-1} (-1)^i f_i^+ = 1.$$

Solution. Since H passes through the interior of P and does not contain any vertex, if it intersects a k-dimensional face of P, then it cuts it into two k-dimensional parts, and the intersection is $(k-1)$-dimensional. Let P_0^+ be the part of P lying in the closed half space $H^+ \cup H$. Then P_0^+ is an n-dimensional convex polytope with the $(n-1)$-dimensional polytope $P \cap H$ as a facet. We denote the numbers of the i-dimensional faces of this polytope $(0 \leq i \leq n-2)$ by g_i, and count the i-dimensional faces of P_0^+.

(1) the faces of P contained in H^+. Their number is f_i^+.

(2) The parts, in $H^+ \cup H$, of the i-dimensional faces of P dissected by H. Since these faces are in bijection with the $(i-1)$-dimensional faces of $H \cap P$, their number is g_{i-1} if $i \geq 1$, and zero if $i = 0$.

(3) The i-dimensional faces of $P \cap H$. Their number is g_i if $i \leq n-2$, and zero if $i = n-1$.

(4) The polytope $H \cap P$ itself is an $(n-1)$-dimensional face of a P_0^+.

Since P_0^+ is an n-dimensional polytope, we can apply Euler's theorem for it. By this,

$$(f_0^+ + g_0) - (f_1^+ + g_0 + g_1) + (f_2^+ + g_1 + g_2) - \ldots + (-1)^{n-2}(f_{n-2}^+ + g_{n-3} + g_{n-2}) + (-1)^{n-1}(f_{n-1}^+ + g_{n-2} + 1) =$$

$$\left(\sum_{i=0}^{n-1} (-1)^i f_i^+ \right) + (-1)^{n-1} = 1 + (-1)^{n-1},$$

which implies the assertion.

Exercise 2. Let $P \subset \mathbb{R}^n$ be an n-dimensional convex polytope, and let $f : \mathbb{R}^n \to \mathbb{R}$ be a linear functional with mutually different values at the vertices of P. For any vertex x let f_i^x P denote the number of the i-dimensional faces F of P that satisfy $f(x) = \max\{f(y) : y \in F\}$. Prove that

$$\sum_{i=0}^{n-1} (-1)^i f_i^x = \begin{cases} 1 & \text{if } f(x) \text{ is the minimum of } f \text{ on } P, \\ (-1)^{n-1} & \text{if } f(x) \text{ is the maximum of } f \text{ on } P, \\ 0 & \text{otherwise .} \end{cases}$$

Solution.
If $f(x)$ is the minimum of f on P-n, then $f_0^x = 1$ and $f_i^x = 0$ for all $i > 0$. Thus, in this case $\sum_{i=0}^{n-1} (-1)^i f_i^x = 1$.

Now, let $f(x)$ be the maximum of f on P. According to our conditions, for any face F containing x it is satisfied that f is maximal on F at x, and thus in this case we need to count all faces of P

containing x. Let H be a level surface of f that strictly separates x from any other vertex of P. Then H intersects exactly those faces of P that contain x (apart from x), and its intersection with a k-face containing x is a $(k-1)$-dimensional face if $k \geq 1$. Since $Q = H \cap P$ is an $(n-1)$-dimensional polytope, and the number of its i-dimensional faces is f_{i+1}^x, Euler's theorem yields

$$1 + (-1)^{n-2} = \sum_{i=0}^{n-2} (-1)^i f_{i+1}^x = -\sum_{i=1}^{n-1} (-1)^i f_i^x.$$

But $f_0^x = 1$, therefore $(-1)^{n-1} = \sum_{i=0}^{n-1} (-1)^i f_i^x$.

Now, assume that $f(x)$ is neither the minimum nor the maximum of f on P. Let H be the level surface of f through x. Consider the $(n-1)$-dimensional polytope $Q = H \cap P$. As x is a vertex of Q there is a 'supporting hyperplane' G in H (i.e. an $(n-2)$-dimensional supporting affine subspace) of Q satisfying $G \cap Q = \{x\}$. Then H can be rotated (by a sufficiently small angle) around G such that f is maximal at x on $P \cap H'$, where H' is the hyperplane obtained by the rotation, but during the rotation the hyperplane does not pass through any vertex of P but x.

Let H'_- be the closed half space bounded by H' that contains all vertices of P at which f attains a smaller value than $f(x)$. Let $P' = P \cap H'_-$ and $Q' = P \cap H'$. Then, for any $2 \leq k \leq n-1$, the $(k-1)$-dimensional faces of Q' containing x are in bijection with the k-dimensional faces of P that contain x and on which f is neither minimal nor maximal at x, and also with the k-dimensional faces of P' that contain x and are not k-dimensional faces of P. Furthermore, f is maximal at x on P', and, in particular , on Q'.

Now we count the faces of P' containing x. Let g_i denote the number of i-dimensional faces of Q' containing x, and let $g_{n-1} = 1$. Then $g_0 = f_0^x = 1$, and every edge of P' that contains x either lies in Q', or it is an edge of P on which f is maximal at x, implying that their number is equal to $f_1^x + g_1$. Finally, if $1 < i \leq n-1$, then every face of P', obtained from a face of P by dissection by H', corresponds to a face of Q' with one less dimension, and hence, in this case the number of i-dimensional faces of P' on which f is maximal at x is equal to $f_i^x + g_i + g_{i-1}$. By applying the result of the previous case to the polytopes P' and Q', we obtain that

$$(-1)^{n-1} = f_0^x - (f_1^x + g_1) + (f_2^x + g_2 + g_1) - \ldots + (-1)^{n-1}(f_{n-1}^x + g_{n-1} + g_{n-2}) =$$

$$\left(\sum_{i=0}^{n-1} (-1)^i f_i^x \right) + (-1)^{n-1},$$

which implies the statement.

Exercise 3. Let $P \subset \mathbb{R}^n$ be an n-dimensional convex polytope, and let F be a k-dimensional face of P. Let $f_j(F, P)$ denote the number of the j-dimensional faces of P containing F. Prove that

$$\sum_{j=k}^{n-1} (-1)^j f_j(F, P) = (-1)^{n-k-1}.$$

Solution.

First, assume that $k = 0$, that is, F is a vertex $\{x\}$. Then there is a linear functional $y \mapsto \langle y, u \rangle$ which is maximal at x on P.By varying the vector u we may assume that this linear functional attains mutually different values at the vertices of P. Then we may aply the second statement from the previous exercise, which implies that if $k = 0$, then $\sum_{j=0}^{n-1} (-1)^j f_j(F, P) = (-1)^{n-1}$.

Now, assume that $k > 0$. Without loss of generality, let $o \in F$, and let X be the orthogonal complement of $\mathrm{aff}(F)$. Let $p : \mathbb{R}^n \to X$ denote the orthogonal projection onto X. Then $p(P)$ is a convex polytope with $p(F)$ as a vertex. Legyen G be an m-dimensional face of P containing

F with $m > k$. If H is a supporting hyperplane of P satisfying $F \subset H$, then by the properties of orthogonal projection, $p(H)$ is a supporting hyperplane of $p(P)$ in X. Thus, $p(G)$ is a face of $p(P)$. Since $\text{aff}(p(G)) = p(\text{aff}(G))$, the dimension of $p(G)$ is $m - k$. On the other hand, let H_0 be a supporting hyperplane of $p(P)$ in X which intersects $p(P)$ in an s-dimensional face containing o. Then $H_0 + \text{aff}(F)$ is a supporting hyperplane of P containing F. We show that this implies that the intersection of $H_0 \cap \text{aff}(F)$ with P is of dimension $s + k$. This is the consequence of the following lemma.

Lemma.
Let $K \subset \mathbb{R}^n$ be a compact, convex set, and let L be a k-dimensional linear subspace. If $K \cap L$ is k-dimensional, and the projection of K onto the orthogonal projection of L is $(n-k)$-dimensional, then K is n-dimensional.

Proof. If $L \cap K$ is k-dimensional, then there are affinely independent points $p_1, \ldots, p_{k+1} \in L \cap K$ in it. Similarly, if the projection of K onto $L^\perp$ is $(n-k)$-dimensional, then there are affinely independent points $q_1, \ldots, q_{n-k+1} \in L^\perp$ contained in the projection of K.By the latter property, The preimages of the points q_j can be written in the form $q_j + x_j$, where $x_j \in L$. We show that the affine hull of the points p_i and $q_j + x_j$ is $\mathbb{R}^n$. For contradiction, assume that there is a nondegenerate linear functional $f : \mathbb{R}^n \to \mathbb{R}$ which attains the same value at every p_i and $q_j + x_j$. Assume that this value is $\alpha \in \mathbb{R}$. As every point of L can be written as an affine combination of the points $p_1, \ldots, p_{k+1}$, it follows that $f(x) = \alpha$ for every $x \in L$. But $o \in L$, from which $\alpha = 0$. Thus, for every j, we have $f(q_j + x_j) = f(q_j) + f(x_j) = 0$, implying $f(q_j) = 0$ for every j. But the points $q_1, \ldots, q_{n-k+1}$ are affinely independent in $L^\perp$, which yields that $f(x) = 0$ for every $x \in L^\perp$. Every point of $\mathbb{R}^n$ can be written in the form $x_1 + x_2$, where $x_1 \in L$, $x_2 \in L^\perp$. Thus, $f(x) = 0$ for all $x \in \mathbb{R}^n$, which contradicts the choice of f. □

The above statement implies that for every s-dimensional face of $p(P)$ containing o can be uniquely assigned to an $(s+k)$-dimensional face of P containing F, and vice versa. Hence, the number of m-dimensional faces of P containing F coincides with the number of $(m-k)$-dimensional faces of $p(P)$ containing o. Thus, the statement follows from the special case $k = 0$ proved in the first part of the solution.

Exercise 4. The f-vector of an n-dimensional convex polytope is $(f_0, f_1, \ldots, f_{n-1}, 1) \in \mathbb{R}^{n+1}$, where f_i denotes the number of the i-dimensional faces of the polytope. Show that the affine hull of the set of the f-vectors of all 3-dimensional polytopes is a plane, or in other words, apart from Euler's formula, there is no other nontrivial linear dependence relation between the face numbers holding for every 3-dimensional polytope.

Solution.

Assume that the equality $\alpha v + \beta e + \gamma f = \delta$ is satisfied for every 3-dimensional convex polytope with v vertices, e edges and f faces. The f-vectors of the five Platonic solids are $(4,6,4,1)$, $(8,12,6,1)$, $(6,12,8,1)$, $(20,30,12,1)$ and $(12,30,20,1)$. Substituting their coordinates into the above equation and solving the system of equations obtained in this way yields that the solution is $\beta = -\alpha$, $\gamma = \alpha$ with $\alpha \in \mathbb{R}$ arbitrary.