Convex Geometry tutorial
for students with mathematics major

Problem sheet 5 - Supporting hyperplanes, faces of convex sets, extremal and exposed points, the Krein-Milman Theorem

Exercise 1. Prove that any compact, convex set in $\mathbb{R}^n$ can be written as the intersection of closed balls.

Exercise 2. Let $K \subset \mathbb{R}^n$ be a compact set. We have shown that if K is convex, then it is supported at every boundary point by a hyperplane. Can this statement be reversed; e.g. if K is supported at every boundary point by a hyperplane, then K is convex?

Exercise 3. Let $K \subset \mathbb{R}^n$ be a compact, convex set, and let F be a face of K. Prove that if p is an extremal point of F, then p is an extremal point of K.

Exercise 4. Let K be a compact, convex set, and let $p \in K$ be a point for which $||p|| \geq ||q||$ for any $q \in K$. Prove that then $p \in \operatorname{ex} K$.

Exercise 5. Let $A \subset \mathbb{R}^n$ be compact. Verify that $p \in A$ is an extremal point of $\operatorname{conv}(A)$ if and only if $p \notin \operatorname{conv}(A \setminus \{p\})$.

Exercise 6. Prove that every exposed point is also an extremal point.