Final exam problems

for the course Applied Numerical Methods with Matlab, 2021/22 spring semester

Students do not need to solve the problems completely at the exam but they must be able to outline the solution process and must know the necessary Matlab functions.

PROBLEM 1. Compute the 1, 2 and ∞ norms of the vector $\overline{\mathbf{x}} = [1, -2, 1, 3, -4]^T$ with Matlab and manually.

PROBLEM 2. Compute the 1, ∞ and 2-norms of the matrix with Matlab and manually (only the first two)

$$\mathbf{C} = \begin{bmatrix} 2 & -1 & 1 & 1 & 1 \\ -1 & 2 & -4 & 2 & 6 \\ -3 & 3 & 2 & -5 & 4 \\ 6 & 7 & -1 & 4 & -9 \\ 4 & -3 & 0 & -3 & 3 \end{bmatrix}.$$

PROBLEM 3. Give an upper bound for the spectral radius of the matrix

$$\mathbf{A} = \begin{bmatrix} 0.5 & 0.5 \\ -0.5 & 0.25 \end{bmatrix}.$$

Is it true that $\varrho(\mathbf{A}) < 1$? Give the sum of the series (if it exists) $\mathbf{I} + \mathbf{A} + \mathbf{A}^2 + \ldots$.

PROBLEM 4. Let us consider the two sequences generated by the iterations

$$x_{k+1} = 2 - 2 \cdot \frac{2 - x_k}{4 - x_k^2}, \qquad y_{k+1} = y_k - \frac{y_k^2 - 2}{y_k + y_{k-1}}$$

where $x_1 = 1$, and $y_1 = 1$, $y_2 = 2$. Both sequences tend to $\sqrt{2}$. Determine the order of the convergence of the sequences.

PROBLEM 5. Compute the value of the expression $((1 + x) - 1)/x$ with Matlab for $x = 10^{-15}$. Explain the result. How can we improve the result?

PROBLEM 6. How would you compute the values of the expressions $\sqrt{1+x} - \sqrt{1-x}$ (for values close to zero) and $\log\sqrt{x+1} - \log\sqrt{x}$ (for large values)?

PROBLEM 7. Show that the matrix

$$\mathbf{A} = \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}$$

is an M-matrix. Give an upper bound for its inverse in maximum norm.

PROBLEM 8. Show that the matrix in Problem 7 is symmetric positive definite.

Problem 9. Give an upper bound in maximum and 1-norms for the condition number of the matrix $\mathbf{A}$ in Problem 7. Compute the condition numbers with Matlab.

Problem 10. Let

$$\mathbf{A} = \begin{bmatrix} 3 & -1 & -1 & 0 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ 0 & -1 & -1 & 3 \end{bmatrix}$$

and let $\tilde{\mathbf{A}}$ be an arbitrary 4×4 matrix such that $\max_{i,j} |a_{ij} - \tilde{a}_{ij}| \leq 0.02$. Give an upper estimate for the relative change of the solution of the system $\mathbf{A}\overline{\mathbf{x}} = [1, 1, 1, 1]^T$ if we change the matrix $\mathbf{A}$ to $\tilde{\mathbf{A}}$. Let us suppose that we know that $\|\mathbf{A}^{-1}\|_\infty = 2.5$. Check the result in Matlab. Give a similar estimate if the elements of the right-hand side vector is allowed to change with values $|\delta b_i| \leq 0.03$!

Problem 11. Solve the system

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 4 & 4 \\ -1 & 5 & 8 \end{bmatrix} \overline{\mathbf{x}} = \begin{bmatrix} 4 \\ 14 \\ 17 \end{bmatrix}$$

with the Gaussian elimination method. Give the LU decomposition of the coefficient matrix.

Problem 12. Solve the system

$$\begin{aligned} 10^{-5}x + y &= 1 \\ x + y &= 2 \end{aligned}$$

with the Gaussian method without and with partial pivoting using decimal floating point numbers with four-digit-long mantissas. Compare the results. Give the general LU decomposition of the coefficient matrix in exact arithmetic.

Problem 13. The LU decomposition of the matrix

$$\mathbf{A} = \begin{bmatrix} 4 & 2 & 1 \\ 2 & 17 & 13/2 \\ 1 & 13/2 & 23/2 \end{bmatrix}$$

is given in concise form

$$\begin{bmatrix} 4 & 2 & 1 \\ 1/2 & 16 & 6 \\ 1/4 & 3/8 & 9 \end{bmatrix}.$$

Give the Cholesky decomposition of $\mathbf{A}$ if it exists.

Problem 14. Give the Cholesky decomposition of the matrix $\mathbf{A}$ in Problem 13 directly, that is without computing the LU decomposition.

Problem 15. Solve the system $\mathbf{A}\overline{\mathbf{x}} = [3, -9/2, -21/2]^T$ ($\mathbf{A}$ is the matrix from Problem 13) using the Cholesky decomposition of the matrix.

Problem 16. Solve the system

$$\begin{bmatrix} -1 & 5 & -2 \\ 1 & 1 & -4 \\ 4 & -1 & 2 \end{bmatrix} \overline{\mathbf{x}} = \begin{bmatrix} 5 \\ 1 \\ -1 \end{bmatrix}$$

with an appropriate (!) iterative solver. Write the computer code in Matlab. Estimate the number of the necessary iteration steps if we would like to achieve an error of 10^{-9} in maximum norm and the iteration starts at $\overline{\mathbf{x}}_0 = [0,0,0]^T$.

PROBLEM 17. Which iterative solvers can be used to solve the system

$$\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \overline{\mathbf{x}} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}?$$

Compute the optimal choice for the relaxation parameter ω in the JOR method. Compare the number of iterations needed to achieve an error of 10^{-6} in maximum norm with the optimal ω parameter and with $\omega = 0.01$ (we start the iteration from the zero vector).

PROBLEM 18. Give a Householder reflection matrix to the vector $\overline{\mathbf{x}} = [2,6,-3]^T$. Give a Givens rotation matrix to the 2:3 subvector of $\overline{\mathbf{x}}$.

PROBLEM 19. Give the QR decomposition of the matrix

$$\mathbf{A} = \begin{bmatrix} 0 & 0 \\ 1 & 3 \\ 0 & 2 \end{bmatrix}$$

with Givens rotations. Compare the result with the result of the command `qr(A)`.

PROBLEM 20. Solve the over-determined system

$$\begin{aligned} 0x + 0y &= 3 \\ x + 3y &= 4 \\ 2y &= 1 \end{aligned}$$

with the `mldivide` command, with the solution of the normal equation, and with QR decomposition.

PROBLEM 21. By the use of Gershgorin's theorem give estimations for the eigenvalues of the matrix

$$\mathbf{A} = \begin{bmatrix} 4.2 & -0.1 & 0.2 & 0.1 \\ 0.05 & 3 & -0.1 & -0.05 \\ 0.5 & -0.5 & 2 & 0.1 \\ 0.1 & 0.2 & 0.3 & -1 \end{bmatrix}.$$

Give an estimation for the spectral radius of the matrix.

PROBLEM 22. Compute the strictly (or single) dominant eigenvalue and the eigenvector of the matrix

$$\mathbf{A} = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$

using the power method with the starting vector $\overline{\mathbf{x}}_0 = [6,7,3]^T$. Compute the eigenvalue (and the eigenvector) closest to 1 starting from the vector $[3,-3,-5]^T$. Solve the problems manually (two iteration steps are enough) and by the use of the provided Matlab code. What is the second largest eigenvalue in absolute value?

PROBLEM 23. Let the matrix $\mathbf{A}$ be defined by the Matlab command `toeplitz([20:-1:1])`. Approximate all the eigenvalues of the matrix with the QR iteration. Give the error of the approximations after 100 steps.

PROBLEM 24. In 1225, Fibonacci gave the only real zero of the polynomial $p(x) = x^3 + 2x^2 + 10x - 20$ to 9 decimal places as $x^\star = 1.368808107$. The exact value is

$$x^\star = \frac{1}{3}\left(\sqrt{27}\sqrt{5240} + 352\right)^{1/3} - \frac{26/3}{\left(\sqrt{27}\sqrt{5240} + 352\right)^{1/3}} - \frac{2}{3}.$$

Show that p has exactly one zero in the interval [0,1]. Approximate the zero with the

a) bisection method (only a few steps, how many step we need to achieve an error of 10^{-6}?),

b) Newton's method (choose an appropriate starting value, give an error estimate after the fourth step, observe the order of the convergence),

c) fixed point iteration using the fixed point reformulation

$$x = \frac{20}{x^2 + 2x + 10}$$

(check the assumptions of the Banach fixed point theorem, observe the order of the convergence).

d) with the `fsolve` command of Matlab.

PROBLEM 25. Find a solution of the system

$$\begin{aligned} x^2 + 2y^2 - y - 2z &= 0 \\ x^2 - 8y^2 + 10z &= 0 \\ x^2 - 7yz &= 0 \end{aligned}$$

near the point $(1, 1, 1)$. Use Newton's method and Matlab's built-in solver.

PROBLEM 26. Construct the interpolation polynomial to the points (-1,6), (0,3), (1,2) with Lagrange's and Newton's methods.

PROBLEM 27. We interpolate the function $f(x) = \ln(x+1)$ on the nodes 0, 0.6, 0.9. Give an upper bound for the interpolation error at the point $x = 0.45$.

PROBLEM 28. We interpolate Runge's function $f(x) = 1/(1+x^2)$ in the interval $[-1, 1]$ on 12 Chebyshev nodes. Estimate the interpolation error (use Matlab to compute and estimate the derivatives of the function).

PROBLEM 29. Calculate the approximate first derivative of the function $f(x) = 1/x$ at the point $x_0 = 0.05$ using the forward difference formula with mesh sizes $h_1 = 0.0016$ and $h_2 = 0.0008$. Use Richardson extrapolation to give a better estimate for the derivative.

PROBLEM 30. Show that the forward finite difference formula

$$D = \frac{-3f_0 + 4f_1 - f_2}{2h}$$

approximates the first order derivative with convergence order 2. Check the result on the function of the previous problem. Use Richardson extrapolation to give a better estimate for the derivative.

PROBLEM 31. The values of a function f at 6 equidistant nodes on the interval $[0, 2\pi]$ are $0, 1, 1, 0, -1, -1$, respectively. a) Calculate the coefficient of $\sin x$ in the trigonometric interpolation polynomial manually. b) Use Matlab's `fft` command to calculate the complex and real discrete Fourier coefficients.

PROBLEM 32. Calculate the Newton–Cotes coefficients $N_{\mathrm{c}}^{4,k}$ and give an approximate value to the integral

$$\int_0^{\pi} \sin x \, \mathrm{d}x$$

with them. Use Matlab's `integral` command to calculate the integral.

PROBLEM 33. Let us apply the composite Simpson's rule to estimate the integral of the previous problem. Let us verify the order of the convergence of the method using Matlab. How small should the length of the subintervals be to achieve an error less than 10^{-6}?

PROBLEM 34. Let us construct the four-point Gauss-Chebyshev formula on the interval $[-1, 1]$! What is the order of the exactness of the method? Verify the result with Matlab. (The nodes are the zeros of the 4th degree Chebyshev polynomial and all weights are $\pi/4$.)

PROBLEM 35. Solve the initial value problem $y' = \mathrm{arctg} y, y(0) = 1$ with the explicit Euler method on the interval $[0, 1]$. Give an estimate to the step size to guarantee an error below 10^{-4}. (The error formula is

$$\|\overline{\mathbf{e}}_k\| \leq e^{(x_{\max}-x_0)L} h(x_{\max} - x_0)M_2/2 \quad .)$$

PROBLEM 36. Solve the IVP

$$y'(x) = \begin{bmatrix} -4 & 5 & 0 \\ 1 & -4 & 1 \\ 0 & 1 & -4 \end{bmatrix} y(x), \; y(0) = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

on the interval $[0, 3]$ using the Crank-Nicolson method. Let the step size be $h = 0.1$.

PROBLEM 37. Solve the initial value problem $y' = x^2(1 + y), y(0) = 3$ with the RK4 method. Verify the order of the convergence of the method. Repeat the investigation with the Heun method.

PROBLEM 38. The Butcher tableau below contains the coefficients of Kutta's third order method.

$$\begin{array}{c|ccc} 0 & 0 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 \\ 1 & -1 & 2 & 0 \\ \hline & 1/6 & 2/3 & 1/6 \end{array}$$

Construct the method from the tableau and apply it to the IVP

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}' = \begin{bmatrix} y_2 + x + 1 \\ -y_1 + x - 1 \end{bmatrix}, \; \begin{bmatrix} y_1(0) \\ y_2(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \; x \in [0, \pi].$$

Let us solve the problem also with the built-in ode45 command. The exact solution is

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} \sin x + x \\ \cos x - x \end{bmatrix}.$$

PROBLEM 39. Consider the following problem of reaction kinetics on the interval [0,50], where we use the initial values 1,1,0, respectively. Solve the problem with Matlab's `ode45`, `ode23s` functions and with the implicit Euler method. Compare the results.

$$\begin{aligned} c_1' &= -0.013c_1 - 1000c_1c_3 \\ c_2' &= -2500c_2c_3 \\ c_3' &= -0.013c_1 - 1000c_1c_3 - 2500c_2c_3 \end{aligned}$$

PROBLEM 40. Using the shooting method, solve the boundary value problem $y'' = y + 4e^x, y(0) = 1, y(1/2) = 2e^{1/2}$! Solve the initial value problems with the Heun method, and apply the bisection method to solve the nonlinear equations. (Exact solution: $y(x) = (2x+1)e^x$.)