BSC IN MATHEMATICS FINAL EXAM

ALGEBRA, ANALYSIS, DISCRETE MATHEMATICS, GEOMETRY

ALGEBRA

1. Number theory

Divisibility, unit, definition of greatest common divisor. Quotient-remainder theorem, Euclidean algorithm. Irreducible number, prime number, fundamental theorem of number theory. Definition of congruence, elementary properties, residue class, complete and reduced residue system. Complex numbers, polynomials, the fundamental theorem of algebra.

2. Systems of linear equations

Homogeneous and inhomogeneous system of linear equations, matrix rank, condition of solvability, number of solutions. R^n, vector space, subspace, linear independence, linear dependence, basis, dimension. Matrix row, column and null space. Gaussian elimination, elementary row operations, row echelon form, reduced row echelon form. Matrix operations and their properties, matrix inverse, transpose.

3. Linear transformations

Concept of linear map, matrix form, change of basis. Eigenvalue, eigenvector, diagonalization. The Euclidean scalar product and norm. Orthogonal and symmetric matrices. Definiteness of quadratic forms. Scalar product over complex numbers and finite fields. Bilinear forms, standard form, principal axis theorem.

4. Matrix decompositions

Jordan's normal form, Cayley – Hamilton theorem. Matrix decompositions: LUP, SVD, QR, spectral decomposition. The main steps of decomposition procedures and the connection among them.

5. Group theory

Group, subgroup, normal subgroup, factor group, homomorphism, homomorphism theorem. Order of a group, of a group element, Lagrange’s theorem, Cauchy’s theorem. Subgroups of a cyclic group. Fundamental theorem of finite Abelian groups, p-groups, Sylow subgroup, Sylow theorems.

6. Polynomial rings, fields

Ring, subring, ideal, factor ring. Field, characteristic, finite fields. In F [x] and Z: division with remainder, principal ideal ring, maximum ideals and factors taken with them. Degree of field extension, multiplication theorem, simple algebraic extension in a given field extension, and their construction as a factor ring of a polynomial ring. Transcendent extension. Finite fields.

ANALYSIS

1. Differentiation

Differentiability of univariate and multivariate functions. Partial and directional derivative. Derivation rules. Inverse function theorem, implicit function theorem. Search for extrema. Techniques for finding a primitive function.

2. Integration of univariate and multivariate functions

Jordan measure, Riemann integral. Lebesgue measure, measurable functions, Lebesgue integral. Beppo-Levi’s theorem, Fatou's lemma, Lebesgue dominated convergence theorem, Fubini's theorem. L_p spaces. Line integral, surface integral. Integral transformation theorems.

3. Complex analysis

Complex differentiability, its relation to bivariate differentiability. Properties of differentiable functions. Complex line integral. Cauchy's theorems and their consequences. Classification of singularities. Residue, residue theorem. Laurent series of a complex function.

4. Series

Numerical series. Power series, radius of convergence, Cauchy-Hadamard theorem. Sequences of functions and series of functions, pointwise and uniform convergence, and consequences. Taylor series expansion with error term. Taylor series of common functions. Functions that can be decomposed into Taylor series. Fourier series: convergence theorems.

5. Metric spaces, normed spaces

Topology of metric spaces, complete metric spaces, Banach fixed-point theorem. Continuous functions in metric spaces, uniform continuity. Compactness in metric spaces. Sequences of continuous functions in metric spaces, pointwise and uniform convergence. Normed spaces.

GEOMETRY

1. Isometries

- classification of isometries on the plane
- classification of isometries in the space
- matrix representation of isometries with inhomogeneous and homogeneous coordinates (reflection, rotation, translation)

2. Non-Euclidean geometries

- models of the hyperbolic and spherical geometry, connection between models (stereographic projection and inversion)
- distance and angle in non-Euclidean geometries (cross ratio)
- internal angle sum of triangles and area in hyperbolic and spherical geometry

3. Differential geometry of curves and surfaces in 3-dimensional Euclidean space

- The fundamental theorem of curve theory, the fundamental theorem of surface theory, Theorema Egregium and the description of notions included.

DISCRETE MATHEMATICS AND ALGORITHMS

1. Data sorting methods. Basic search methods, their data structures

Selection, insertion, quick, merge sort, lower bound for the number of comparisons in sorting algorithms, bin sort, radix sort. Binary tree traversals, binary search trees, red-black-trees, 2-3-trees.

2. Shortest paths in graphs

Breadth-first search (BFS), Dijkstra, Ford and Floyd Algorithms.

3. Search for minimum-weight spanning trees in graphs, search for maximum matching in bipartite graphs, flows.

Concept of minimum-cost spanning trees, greedy algorithm, Kruskal theorem. Concept of bipartite graphs, matching in bipartite graphs, Hall and Frobenius theorem, alternating path algorithm. Network flows, Ford-Fulkerson theorem, augmenting path algorithm.

4. The notion of NP, famous problems in NP. NP completeness

The NP language class, Karp reduction, NP completeness, Cook-Levin theorem, famous NP-complete languages: 3SAT,HAM-CYCLE, HAM-PATH, 3COLOR, MAXCLIQUE, Max Independent Set, SUBSETSUM, PARTITION
(Linear) integer programming (IP), translation of combinatorial problems to integer programming.

5. Colorings of graphs, graph traversals, planar graphs

Colorings of graphs, chromatic number, clique number, lower and upper bounds on chromatic number. Edge chromatic number, Vizing theorem, Kőnig theorem. BFS, DFS, DAG. Planar graphs, Euler formula, four-color theorem, homeomorphism (topological isomorphism), Kuratowski theorem, Fáry-Wagner theorem.