Small zero divisors in maximal orders of $M_n(\mathbb{Q})$

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Abstract

The topic of this thesis stems from the representation theory of finite dimensional algebras: let $\mathcal{A}$ be an associative algebra given by structure constants over an algebraic number field $\mathbb{K}$ which is isomorphic to the full matrix algebra $M_n(\mathbb{K})$; we would like to construct explicitly an isomorphism $\mathcal{A} \to M_n(\mathbb{K})$. We consider the case $\mathbb{K} = \mathbb{Q}$. In order to construct this isomorphism, we need to find a rank 1 matrix with a small Frobenius norm in the maximal $\mathbb{Z}$-order in $\mathcal{A}$, where the Frobenius norm is inherited from an embedding of $\mathcal{A}$ into $M_n(\mathbb{R})$.

In a recent paper G. Ivanyos, L. Rónyai and J. Schicho proved that there exists such a matrix whose Frobenius norm is less than n. The upper bound for the norm which follows from the proof is in fact stricter: it is the Hermite constant γ_n. Thus a naturally arising question is whether this upper bound is optimal, i. e. if there is a regular real matrix P such that the minimal Frobenius norm of PAP^{-1} for all rank 1 matrices $\mathcal{A}$ is γ_n.

Besides being interesting on their own this topic has a number of connections with important problems in algebraic geometry and number theory, including the very famous Birch and Swinnerton-Dyer conjecture.

We use lattices in Euclidean spaces to examine this problem, thus establishing a firm link between the theory of lattices and the representation theory of algebras. After summarizing the background of the problem, prove that although Hermite's constant is a good upper bound for the norm of rank one matrices in maximal orders, the tight upper bound is the so-called Bergé-Martinet constant. Then we describe the lattices attaining the Bergé-Martinet constant.

Furthermore, we consider the case of general singular matrices. After realizing that we have been working with tensor products of lattices, we use the results of Steinberg and Kitaoka to decide whether the matrix with the smallest Frobenius norm in a maximal order always has rank one. The answer will be negative in general, but affirmative in small dimensions. We deduce a stronger version of Kitaoka's theorem in dimensions at most 8. Using these results we present a substantially improved version of the first algorithm of IRS in small dimensions. Our variant surpasses the original method in two ways: it has a simpler control structure as it no longer involves jumps, moreover, a better bound is given for the size of the region to be searched. Finally we outline a possible generalization of the problem to algebras over other algebraic number fields.