Feynman Path Integrals and their Discrete Approximations

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Abstract

Feynman's path integral formulation is the generalization of the classical action principle in quantum mechanics. In 1948 Richard Feynman proposed that if we take Schrödinger's equation for one particle:

$$i\hbar\frac{\partial\Psi(t,x)}{\partial t} = -\frac{\hbar^2}{2m}\Delta\Psi(t,x) + V(x)\Psi(t,x), \quad \Psi(0,x) = g(x),$$

then the solution can be given in the form of the integral on the continuous trajectories:

$$\Psi(t,x) = \frac{1}{Z}\int \exp\left(i\int_0^t \frac{1}{2}\left(\frac{d\omega(s)}{ds}\right)^2 - V(\omega(s))\,\mathrm{d}s\right) g(\omega(0))\mathcal{D}\omega.$$

This integral can be interpreted as the weighted sum of all the continuous trajectories the particle could take if it was a classical point mass. This formulation has furhter generalizations in quantum physics (like quantum field theory), and provides to be a greatly useful tool in that area.

Despite this, mathematical problems arise around it. The path integral contains objects which cannot exist mathematically, i.e. the Lebesgue-type integrating measure on the space of continuous trajectories, and also the normalizing factor cannot have a definite value.

On the other hand, if the almost identical real-valued equation

$$\frac{\partial\Psi(t,x)}{\partial t} = \frac{1}{2}\Delta\Psi(t,x) - V(x)\Psi(t,x), \quad \Psi(0,x) = g(x)$$

is considered, then the resulting similar path integral can be given a rigorous mathematical meaning, as Mark Kac showed. The integral in this case is called the Feynman–Kac formula.

There is a rich literature about giving a solid mathematical foundation for Feynman's path integral. An approach based on measure theory has not been given so far. In this thesis the aim is to present an approach based on pathwise approximations by random walks first in the real-valued, then in the complex-valued case.

In the real-valued case a sequence of random walks can be constructed, which for each walk is a refinement of the previous one in some sense. Using this construction, a discrete path integral process can be defined, which converges to the continuous path integral almost surely. Furthermore, the expectation of the discrete process solves the discretized version of the real Schödinger-type equation, and converges to the solution of the continuous equation.

The goal is to extend this construction to the complex case. As a first step, random walks with complex transition amplitudes are introduced. Then again, a discrete version of the path integral is introduced, which satisfies the discretized version of the complex Schrödinger's equation.

In the end of the thesis some heuristics and conjectures are shown about this complex random walk. A hypothesized de Moivre–Laplace-type approximation for the complex transition amplitudes is shown. It is also hypothesized that if we take the complex probabilities (complex transition amplitudes squared, and normed with their sum), then they correspond to the real probabilities of a simple, symmetric random walk, i.e. a coupling can be found between the two random walks.