The Iterative Solution of the p-Laplacian Type Problems

Master's thesis

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In the present thesis we investigate the properties of p-Laplacian and p-Laplacian type operators together with its finite element approximation. The p-Laplacian operators and its finite element approximation has been extensively studied in the literature. It arises in several problems in mathematical physics where nonlinear mechanism of diffusion and heat conduction has to be taken into account. It is an example of degenerate nonlinear systems and it exhibit several computational difficulties.

We present the properties of p-Laplacian type operators and examine the finite element approximation of it with Dirichlet data [1], [6], [11]. We will see that the weak solution of the p-Laplacian is equivalent to the minimization of a nonlinear energy functional, which is strictly convex and has a unique minimum. We examine some computational issues on the discretization of the degenerate p-Laplacian, based on [2] and [3]. This is a preconditioned descent algorithm with power order convergence.

We propose a preconditioned quasi-Newton-Kantorovich algorithm for a non-degenerate p-Laplacian type operator [4]. The Newton-Kantorovich method is a direct generalization of the classical Newton method for operators between Banach spaces. We give an iterative solution of the $F(u) = 0$ operator equation in Hilbert space. In the case of the original Newton-Kantorovich method we calculate in each step the following: $u_{n+1} = u_n - F'(u_n)^{-1}F(u_n)$, whereas in the case of the quasi-Newton-Kantorovich method we use variable preconditioning, that is we replace operator $F'(u_n)^{-1}$ in each step with a spectrally equivalent one. First we establish the linear convergence for locally elliptic operators with locally Lipschitz continuous derivatives. Then we will apply this result for a particular p-Laplacian type operator, we give appropriate preconditioning operators and we prove the convergence of the method. Our results are also applicable to other p-Laplacian type operators which satisfy certain Lipschitz continuity conditions. The proposed scheme was implemented and numerical experiments were conducted. We varied the parameters of the system and the method was convergent, which confirmed our our theoretical arguments.

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