Decomposability of multiple planar coverings

Master's thesis

István Kovács
Budapest University of Technology and Economics

Supervisor: Dr. Géza Tóth

The study of multiple coverings was initiated by Davenport and L. Fejes Tóth about 60 years ago [1]. Is it true that if m is large enough, then the covering can be decomposed into two (or more) coverings? This simple question leads to surprisingly hard problems, most of them still unsolved. Besides its theoretical interest, the problem has important practical applications to sensor networks.

A planar set P is said to be cover-decomposable if there is a constant $m = m(P)$ such that every m-fold covering of the plane with translates of P can be decomposed into two coverings.

J. Pach proposed the problem of determining all cover-decomposable sets, in 1980. According to his conjecture, all planar convex sets are cover-decomposable. The conjecture has been verified only in some special cases, in particular, it is known, that all open convex polygons are cover-decomposable. Very recently, D. Pálvölgyi [6] *refuted* Pach's conjecture. He proved that open and closed sets with smooth boundary are not cover-decomposable. In particular, the unit disc is not cover-decomposable. All general positive results so far hold only for open sets.

The thesis contains some of our results in this topic. We show that closed, centrally symmetric convex polygons are also cover-decomposable. We also show that an *infinite-fold* covering of the plane with translates of P can be decomposed into two infinite-fold coverings. Both results hold for coverings of any subset of the plane. See [5].

A *homothetic* transformation is the composition of a translation and a scaling. Keszegh and Pálvölgyi [3] proved that any 12-fold covering of the plane with homothetic copies of a fixed triangle T can be decomposed into two coverings. In this thesis we show that, with a few possible exceptions, this result cannot be extended to other polygons. We prove that for any convex polygon S with at least four sides, or a concave one with no parallel sides, and any $m > 0$, there is an m-fold covering of the plane with homothetic copies of S that cannot be decomposed into two coverings. See [4].

Let $\mathcal{H}$ be a set of half-planes, covering the plane many times. Can we decompose the covering into two (or more) coverings? In the finite case, if there are only finitely many half-planes, the answer is known to be yes [2, 7]. We prove that any 11-fold covering by closed half-planes of the plane is decomposable into two coverings. We prove this result as a consequence of its dual version, that is, first we prove that any planar point set can be colored with two colors such that any closed half-plane which contains at least 6 of the points, contains a point of both colors. We apply similar ideas as before, so in this thesis we only give a sketch of the proof.

References

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