Transformations Preserving the Sandwiched Rényi Entropy and its Generalizations
(Abstract)

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Relative entropy is one of the most important numerical quantities in quantum information theory. It is used as a measure of distinguishability between quantum states, or their mathematical representatives, the density operators. In fact, there are several concepts of relative entropy among which the most common one is due to Umegaki. In [2] my supervisor L. Molnár determined the general form of all bijective transformations on the set of density operators which preserve that type of relative entropy. In the paper [5] the bijectivity assumption was removed from the result of [2] while in [3] the structures of preservers of other types of relative entropy were determined. After this, in [4] a far-reaching generalization of the previously mentioned results was given. Namely, all transformations on the set of density operators which preserve the so-called quantum f-divergence with respect to an arbitrary strictly convex function were determined.

The main results of this thesis are closely related to the aforementioned ones. We were interested in characterizing all transformations on the set of density operators leaving the sandwiched Rényi entropy invariant and determining the structure of all bijective transformations on the cone of positive definite operators which preserve this quantity. Obviously, if this quantity were quantum f-divergence corresponding to a strictly convex (or strictly concave) function, then the result in [4] would apply and we would be done. Therefore, it was necessary to verify that the quantity

$$D'_\alpha(A\|B) = \begin{cases} \operatorname{Tr}\left(B^{\frac{1-\alpha}{2\alpha}} A B^{\frac{1-\alpha}{2\alpha}}\right)^{\alpha}, & \operatorname{supp} A \not\perp \operatorname{supp} B \wedge (\operatorname{supp} A \subseteq \operatorname{supp} B \vee \alpha < 1) \\ \infty, & \text{otherwise.} \end{cases}$$

is not a quantum f-divergence on the set of density operators. (We remark that

$$D_\alpha(A\|B) = (\alpha-1)^{-1} \log\left((\operatorname{Tr} A)^{-1} D'_\alpha(A\|B)\right)$$

is the definition of the sandwiched Rényi entropy in the case where $\alpha \in \,]0,1[\,\cup\,]1,\infty[$, and on the set of density operators the preservation of one of these quantities is equivalent to the preservation of the other one.)

Proposition 1. *For any* $\alpha \in \,]0,1[\, \cup \,]1,\infty[$ *we have that* $D'_\alpha(.||.)$ *is not a quantum f-divergence on the set of density operators.*

We have solved a much more general preserver problem than finding the symmetries of transformations preserving the sandwiched Rényi entropy. In a certain way we have generalized this relative entropy notion and determined the corresponding symmetries on the set of density operators. Naturally, it has given the answer to our first question.

Theorem 2. *If* $\alpha \in \,]0,1[\cup]1,\infty[$ *and* ϕ *is a transformation on the set of density operators satisfying*

$$D_\alpha(\phi(A)||\phi(B)) = D_\alpha(A||B)$$

for every density operators A, B*, then there is either a unitary or an antiunitary operator* U *on* H *such that* ϕ *is of the form*

$$\phi(A) = UAU^*.$$

Finally, we have considered the set of positive definite operators and described the structure of all bijective transformations preserving the sandwiched Rényi entropy. We noticed that a similar result is valid concerning the case of positive operators.

Theorem 3. *If* $\alpha \in \,]0,1[\cup]1,\infty[$ *and* ϕ *is a bijective transformation on the set of positive definite operators satisfying*

$$D_\alpha(\phi(A)||\phi(B)) = D_\alpha(A||B)$$

for every positive definite operator A, B*, then there is either a unitary or an antiunitary operator* U *on* H *and a positive scalar* c *such that* ϕ *is of the form*

$$\phi(A) = cUAU^*.$$

Although the results are similar, we point out that there are significant differences between the proofs of Theorem 2 and 3. These results of ours will be published in [1].

References

[1] M. Gaál and L. Molnár, *Transformations on density operators and on positive definite operators preserving the quantum Rényi divergence and its generalizations,* submitted to Quantum Inf. Process.

[2] L. Molnár, *Maps on states preserving the relative entropy,* J. Math. Phys. **49** (2008), 032114.

[3] L. Molnár and G. Nagy, *Isometries and relative entropy preserving maps on density operators,* Linear and Multilinear Algebra, **60** 2012, 93–108.

[4] L. Molnár, G. Nagy and P. Szokol, *Maps on density operators preserving quantum f-divergences,* Quantum Inf. Process. **12** (2013), 2309–2323.

[5] L. Molnár and P. Szokol, *Maps on states preserving the relative entropy II,* Linear Algebra Appl., **432** (2010), 3343–3350.