Mixing time of the Glauber dynamics for the Ising model

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Abstract

The Ising model is a simplified description of ferromagnetism in statistical physics. The model consists of a graph, its vertices have spins +1 or −1, that represent the magnetic dipole moment of atomic spins. The spins interact with the neighbors: if all neighbors have positive spins, the original point will more likely to have positive spin as well.

The strength of the interaction depends on a parameter β, which is interpreted as the inverse temperature. Like the Curie point in reality, the inverse temperature parameter also has a critical value in the model, where transition between ferromagnetic and paramagnetic phases occurs. The Glauber dynamics is the simplest Markov-chain on the Ising model, and its mixing time depends on the temperature and the type of the graph for which we are applying the model.

The leitmotif of the thesis is that complicated Markov chains can sometimes be understood via projections onto simpler processes. In the case of the Ising Glauber dynamics, the most natural choice is to consider the total magnetization of the chain, which is simply the sum of the spins in the system. For the complete graph, this integer-valued process is again a Markov chain, whose analysis directly yields results for the full chain, as well. However, the case of the square lattice is much more complicated: because of the non-trivial underlying geometry, the magnetization process is not Markovian anymore, hence it is much harder to understand, and its relation to the full chain is more mysterious. The main goal of this thesis is to shed more light on this relationship.

In the beginning of the thesis we introduced the basic definitions and methods regarding the mixing time of a Markov chain, and in the next chapter we applied these techniques on three examples that can be considered as simplified versions of the dynamics. In the last two chapters we examined the Glauber dynamics on the complete graph and on the square lattice. We summarized previous results, and conducted simulations to get a better understanding of the behavior of the dynamics.

For the complete graph, the results of our simulations were close to the theoretical results and described the limit of the dynamics well. However, for the square lattice our simulation to define an upper bound of the mixing time gave a lower result as we expected, this difference might derive from the fact that the state space for larger graphs is enormous and the simulations have large computation needs to give a more exact result.

Another simulation we conducted on the square lattice was to approximate the relaxation time of the dynamics. Our result in this case approaches the empirical mixing time measured by physicists, which gives a better approximation of the relaxation time than the so far known theoretical results. Moreover, it confirms the conjecture, that while magnetization mixes the most slowly on the square lattice, by the Dirichlet form (that is a common method to define a lower bound of the mixing time) it does not lead to an efficient lower bound.