Crossing numbers of graphs and their relationships

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Abstract

In my BSc thesis, I investigate the relationships between four interesting versions of the crossing number. The *crossing number* of a graph G, $\textsc{cr}(G)$, is the smallest number of edge crossings in any drawing of G. The *rectilinear crossing number* of G, $\textsc{lin-cr}(G)$, is the minimum number of crossings in any drawing of G, in which every edge is represented by a straight-line segment. The *pairwise crossing number* of G, $\textsc{pair-cr}(G)$, is the minimum number of pairs of edges (e, e') such that e and e' determine at least one crossing, over all drawings of G. The *odd-crossing number* of G, $\textsc{odd-cr}(G)$, is the minimum number of pairs of edges (e, e') such that e and e' cross an odd number of times.

We have $\textsc{lin-cr}(G) \geq \textsc{cr}(G) \geq \textsc{pair-cr}(G) \geq \textsc{odd-cr}(G)$. Determining any of these crossing numbers is an extremely difficult problem, all of them are NP-hard.

The rectilinear crossing number is equal to the crossing number whenever the latter is at most 3. However, if the crossing number of a graph is 4, the rectilinear crossing number can be arbitrarily large. On the other hand, $\textsc{pair-cr}(G) \leq \textsc{cr}(G) \leq 2\textsc{pair-cr}(G)^2$, the upper bound $\textsc{cr}(G) \leq 2\textsc{pair-cr}(G)^2$ has been improved in several steps, but it is still unknown whether $\textsc{pair-cr}(G) = \textsc{cr}(G)$ holds for every graph G. It is also true that $\textsc{odd-cr}(G) \leq \textsc{pair-cr}(G) \leq 2\textsc{odd-cr}(G)^2$, there are examples of graphs, where $\textsc{odd-cr}(G) < \textsc{pair-cr}(G)$. In my thesis, I show two such examples. On the other hand, the quadratic upper bound $\textsc{pair-cr}(G) \leq 2\textsc{odd-cr}(G)^2$ has not been improved.

Arguably the most important inequality about the crossing number is the Crossing Lemma, which states that $\textsc{cr}(G) \geq \frac{1}{64}\frac{e^3}{n^2}$ for every graph with n vertices and $e \geq 4n$ edges. This inequality, and in general, the crossing number plays an important role in various fields of discrete and computational geometry, and they can also be used to obtain lower bounds on the chip area required for the VLSI circuit layout of a graph.