The projection of the random Menger sponge

Vilma Orgoványi
Supervisor: Prof. Károly Simon

In this thesis we consider a special three-dimensional Mandelbrot percolation fractal, namely the random Menger sponge. The construction of the random Menger sponge consists of infinite iteration of the following steps. The initial set is the three-dimensional unit cube $[0,1]^3$. The cubes we retained at the n^{th} iteration we subdivide into smaller cubes of the same size, discard those that are not contained in the level $(n+1)$ approximation of the Menger sponge, and each of the remaining subcube is retained with some predefined probability p or discarded, independently of each other. The random Menger sponge consists of those points which we have never discarded. Our aim was to find the size of the projection of the random Menger sponge to the spacial diagonal of the unit cube with different choices of the parameter p. It turned out that any of the following phases are possible:

1. The Hausdorff dimension of the percolation is smaller than one, and in this way, the projection has zero Lebesgue measure almost surely.

2. The Hausdorff dimension of the percolation is greater than one, but the projection has zero Lebesgue measure almost surely.

3. The Hausdorff dimension of the percolation fractal is greater than one, and the projection has positive Lebesgue measure but does not contain an interval almost surely conditioned on non-extinction.

4. The Hausdorff dimension is greater than one and the projection contains an interval almost surely conditioned on non-extinction.

We could not only prove that the random Menger sponge can be in the above phases, but found parameter intervals where it is surely in Phase 1, 3 and 4.