Point Sets with Rational Distances
EXTRACT

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My thesis deals with the problem of point sets in real normed spaces, particularly in the Euclidean plane, which have the property that all pairwise distances are rational or integer numbers. The modern history of the topic started with the results of Paul Erdős and Norman H. Anning, who proved in 1945 that an infinite point set in the Euclidean plane with all integer distances must be entirely collinear. Erdős conjectured that even if we allow rational distances, the possible sets must be very restricted, and in general position they must be finite. Stanislaw Ulam conjectured that no rational distance set can be dense in the Euclidean plane; this is known today as the Erdős-Ulam conjecture and has remained unresolved.

I first summarise the basic facts regarding the cardinality, geometric invariance, coordinate representation and distance-matrix representation of planar rational and integral distance point sets. After my own proofs on their $\aleph_0$ cardinality in finite dimensions, I present Erdős' example of a rational distance set of cardinality $\aleph_1$ in the Hilbert-space ℓ^2. I demonstrate the widely used feature that not only isometries and rational scaling, but also certain circle inversions preserve rational distances. Afterwards I introduce Kemnitz's lemma, stating that rational distance sets can always be represented in the coordinate grid $\mathbb{Q} \times \sqrt{k}\mathbb{Q}$ with some square-free rational k. For finite integral distance sets their distance matrices and the respective Cayley-Menger determinants provide an alternative frame of reference.

In the next section I present the most important constructions of rational and integer distance sets in the plane. Several concyclic and quasi-collinear examples were given by Erdős motivated by number theoretic and geometric considerations. I explain and complete the sketches of his proofs, and also show a straightforward generalisation to build infinite rational distance sets with all but 2 points on a line. Further, I give a brief overview of the best known examples of general position rational distance sets.

The last section deals with restrictive structure theorems. I explain in details the geometrical and combinatorial proofs that Erdős and Anning gave on infinite integral distance sets in the plane, concluding that such sets must be entirely collinear. Finally, I address two recent important theorems by Solymosi and de Zeeuw. With the use of deeper algebraic tools they proved that if a rational distance set has infinitely many points on a line (resp. circle), then it can have at most 4 (resp. 3) points off that line (resp. circle).