Rapid mixing of doubly stochastic Markov chains on the torus

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In this thesis we set out to find the fastest mixing nearest-neighbour Markov chains with a uniform stationary distribution on the d dimensional discrete torus $(\mathbb{Z}/n\mathbb{Z})^d$ with side length $n \in \mathbb{N}^+$, where $d \geqslant 2$. We managed to prove that our non-reversible chain in dimension $d = 3$ is optimal in the sense that its mixing time is in $\Theta(n)$ and the mixing time of any nearest-neighbour Markov chain on the torus must be contained in $\Omega(n)$. With some additional effort, one can generalize our proof to any higher dimensions. Although we could not finish the work in the two dimensional case, we constructed a Markov chain, which is conjectured to have a mixing time in $O(n^{\frac{4}{3}})$ and we provide numerical evidence in favour of this intuition.

The thesis starts out in Section 3 with building up the mathematical toolkit which we use in our proofs. These are the notions and theorems related to mixing times as well as results that connect these definitions to couplings of Markov chains. This section mostly relies on the book *Markov Chains and Mixing Times* by Levin, Peres and Wilmer.

In the next section (Section 4) we investigate how far we can go if we use reversible Markov chains. To this end we introduce spectral gaps and Dirichlet forms with a few related theorems. Next, we show that the Lazy Random Walk has a mixing time in $\Theta(n^2)$. Lastly, we prove that it is among the fastest mixers within the class of doubly stochastic reversible Markov chains on $(\mathbb{Z}/n\mathbb{Z})^d$. In fact, we prove that lazy random walk has the largest spectral gap in this class. Hence we need a non-reversible Markov process if we want to achieve a mixing time in $\Theta(n)$.

In Section 5 we start presenting our findings. Diverging from the torus for a bit we introduce a simple non-reversible Markov chain (the *carousel*) and bound its mixing time. The significance of this sidetrack is twofold. First, it serves as an example on how one can derive mixing time bounds using couplings. Second, this Markov chain is an important building block of our fast mixing strategy on the torus, so we can reuse the calculations shown in this section.

In Section 6 we are able to prove our main result. The long awaited Markov chain is introduced along with more notions and lemmas. As a grand finale we put together the most important results from section 3, 5 and 6 which yields that the mixing time of our chain on $(\mathbb{Z}/n\mathbb{Z})^3$ grows linearly with respect to the side length n.

In Section 7 we explain why our methods fail in the two-dimensional setting and present possible resolutions. Then we describe our best strategy based exclusively on heuristic ideas. As it is not proven yet that this Markov chain has mixing time in $O(n^{\frac{4}{3}})$, we give numerical results which are consistent we our hypothesis. Naturally, we write a few words about the simulations themselves and provide a link to the codes and some illustrative GIFs with the help of QR codes.