Thesis Summary

Title: The Brunn-Minkowski Theory

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In this thesis we investigate one of the fundamental inequalities of convex geometry, the Brunn-Minkowski Inequality, stating that for any compact sets A, B in the n-dimensional Euclidean space $\mathbb{R}^n$ and $\lambda \in [0,1]$, we have

$$\mathrm{vol}((1-\lambda)A + \lambda B)^{1/n} \geq (1-\lambda)\,\mathrm{vol}(A)^{1/n} + \lambda\,\mathrm{vol}(B)^{1/n},$$

where $\mathrm{vol}(\cdot)$ denotes n-dimensional volume. This inequality is the cornerstone of the Brunn-Minkowski theory and a beautiful and powerful tool for solving various difficult problems involving measurement relationships such as surface area, volume, width, etc. The thesis provides two proofs, one of them employs the concept of Hadwiger-Ohmann cut from measure theory, while the other rests on the Prékopa–Leindler Inequality.

As an application of this inequality, we prove the Isoperimetric Inequality, stating that for any convex body $K \subset \mathbb{R}^n$,

$$\left(\frac{S(K)}{S(B_2^n)}\right)^{\frac{1}{n-1}} \geq \left(\frac{\mathrm{vol}(K)}{\mathrm{vol}(B_2^n)}\right)^{\frac{1}{n}},$$

where $S(\cdot)$ denotes surface area and B_2^n denotes the closed Euclidean unit ball in $\mathbb{R}^n$ centered at the origin. We explore the connection of this inequality to mixed volumes, and introduce, and partly prove, some of its generalizations, namely Minkowski's inequalities and the Alexandrov-Fenchel Inequality.

Furthermore, we also prove the Reverse Isoperimetric Inequality, which was originally discovered by Ball. Along the way we study the properties of ellipsoids, and demonstrate John's theorem concerning the maximum volume ellipsoid contained in various types of convex bodies.