Interior angle sums of triangles in Thurston geometries

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Extract

In this paper, we have explored translation triangles within several non-Euclidean Thurston geometries, focusing on how their interior angle sums are influenced by the structure of the space. While in Euclidean geometry the interior angles of a triangle always sum to exactly π, in more complex three-dimensional homogeneous geometries, this sum varies in ways that reflect the curvature, anisotropy, and group-theoretic properties of each studied space.

We concentrated our investigation on the following non-isotropic Thurston geometries: $\mathbb{S}^2 \times \mathbb{R}$, $\mathbb{H}^2 \times \mathbb{R}$, Nil, Sol and $\widetilde{\mathrm{SL}_2\mathbb{R}}$, each of which displays distinct geometric behavior. We used projective models of Thurston geometries in order to investigate triangle interior angle sums. Due to the comlpexity of the geodesic curve we have no exact solution in Sol space yet. Therefore, we have considered translation curves.

These findings demonstrate that the triangle angle sum is a powerful geometric invariant, sensitive not only to curvature but also to the anisotropy and the global geometry of the space. The methods employed—rooted in differential geometry, Lie group theory, and geometric intuition allow us to link local geometric phenomena to the broader classification provided by Thurston's eight geometries.

This thesis offers a foundation for further study in the direction of polygonal geometry in homogeneous spaces, geodesic flows, and discrete group actions. It also enriches the understanding of geometric structures on 3-manifolds, contributing to the intuition and tools needed for navigating Thurston's Geometrization program.

In conclusion, triangles serve as an insightful lens into the geometry of 3-manifolds, and their study within the framework of Thurston geometries reveals the deep interplay between curvature, symmetry, and topology in low-dimensional spaces.