Abstract: Densities of Soft Ball Lattice Packings in the Diagonal Family

An important question of discrete geometry is to find, for any $d \geq 2$, the maximum density of packings of congruent d-dimensional balls; that is, the maximum portion of space, in terms of volume, that can be packed by congruent d-dimensional balls. The goal of this thesis is to investigate a variant of this problem, namely the (soft) density of soft packings of congruent balls. We restrict our investigation to the $d = 3$ case.

Following a paper of Bezdek and Lángi [1,2], we call a family of balls of radius $(1+\lambda)r$ a soft ball packing with penetrating ratio $\lambda > 0$ if the concentric balls of radius r form a packing. As a result of Böröczky [3], if $\lambda \geq \sqrt{\frac{5}{3}} - 1 \approx 0.29$, then it is possible to cover the whole space by such a soft packing. By above result of Böröczky and Hales's proof [3, 6] of the Kepler Conjecture, it seems reasonable to assume that if $\lambda \approx 0$, then a densest soft packing is generated by a face-centered cubic lattice, while if $\lambda \approx \sqrt{\frac{5}{3}} - 1$, then such a soft packing is generated by a body-centered cubic lattice. Thus, as we increase λ from 0 to $\sqrt{\frac{5}{3}} - 1$, the optimal configurations shift between two different type arrangements.

The goal of this thesis is to study the density of soft ball packings generated by a lattice from a special 1-parameter family of lattices, called the diagonal family of lattices. These lattices, which include the cubic, face-centered cubic, and body-centered cubic lattices, were defined by Edelsbrunner and Kerber [5] and used by Edelsbrunner and Iglesias-Ham [4] to study the soft densities of soft ball packings generated by them, employing a definition of soft density different from ours.

Our main result in this thesis states the following: there exists a special value $\lambda_0 \approx 0.199 \in [0, 0.28]$ such that:

(i) for $0 < \lambda \leq \lambda_0$, among the soft ball packings generated by a lattice from the diagonal family, the densest one is generated by a face-centered cubic lattice;

(ii) for $\lambda_0 \leq \lambda \leq 0.028$, among the soft ball packings generated by a lattice from the diagonal family, a Body-centered cubic lattice generates the densest one;

Our proof uses the combinatorial description of the Voronoi cells of the lattices in the diagonal family, given by Edelsbrunner and Iglesias-Ham [4], and some other geometric and analytic tools. The computations included in the proof were carried out by the Maple 18.00 software.

References

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