Abstract

Extremal graph theory studies among other things the maximum size of graphs avoiding certain substructures. A central function is the Turán number $\mathrm{ex}(n, H)$: the maximum number of edges in an n-vertex graph with no subgraph isomorphic to H. This thesis focuses on the Turán numbers of complete bipartite graphs $K_{s,t}$ and the *regular Turán number* $\mathrm{rex}(n, H)$, the maximum number of edges in a *regular* n-vertex H-free graph. We first review classical extremal results, including Mantel's theorem (1907) for K_3, Turán's theorem (1941) for cliques, and the Erdős–Stone theorem (1946). For bipartite H, these results yield only $\mathrm{ex}(n, H) = o(n^2)$, prompting the study of the Zarankiewicz problem. We present the Kővári–Sós–Turán theorem, which gives the upper bound $\mathrm{ex}(n, K_{s,t}) = O(n^{2-1/s})$, and discuss constructions that attain matching lower bounds. In particular, we explore algebraic constructions from finite field theory: the norm graph $\mathrm{G}_{q,s}$ of Kollár–Rónyai–Szabó and its variant, the projective norm graph $\mathrm{NG}(q, s)$ by Alon–Rónyai–Szabó. We include complete proofs that these graphs are $K_{s,t}$-free for appropriate t (e.g. $t > s!$ or $t > (s-1)!$), and that they contain $\Theta(n^{2-1/s})$ edges, thereby solving $\mathrm{ex}(n, K_{s,t})$ asymptotically for those cases.

In the second part, we discuss in detail the regular Turán number $\mathrm{rex}(n, H)$, first studied by Gerbner, Patkós, Tuza, and Vizer (2019). We survey known results for $\mathrm{rex}(n, F)$, including constructions for cycles and complete bipartite graphs. Recent work of Tait and Timmons (2021) is highlighted, showing that for $F \in \{C_4,\ K_{2,t},\ K_{3,3},\ K_{s,t}\}$ (with t large), one can construct regular F-free graphs with edge counts matching the Turán function up to constant factors. Notably, we detail their methods (algebraic and geometric) that yield $\mathrm{rex}(n, C_4) = \Theta(n^{3/2})$, $\mathrm{rex}(n, K_{3,3}) = \Theta(n^{5/3})$, and in general $\mathrm{rex}(n, K_{s,t}) = \Theta(n^{2-1/s})$ for $t > s!$. We then contribute a simplification of some of the proofs using bipartite projective norm graphs and bipartite norm graphs, making regularization easier. The thesis is structured as follows: after the introduction, Chapter 2 covers foundational theorems; Chapter 3 develops the norm graph constructions and their extremal properties; Chapter 4 examines regular Turán numbers and reproduces key proofs from the literature, as well as our new observations; finally, Chapter 5 concludes with open problems and conjectures for future work in this area.