Abstract

Triangle Centers and the Euler Line in Non-Euclidean Geometries

This thesis explores how classical triangle centers, namely the centroid, circumcenter, and orthocente, are defined and behave in hyperbolic geometry, using the Poincaré disk model. In Euclidean geometry, these three centers lie on a single straight line, known as the Euler line.However, in hyperbolic geometry, this property does not automatically hold due to the curved nature of the space.

To investigate these centers, we represent hyperbolic lines and circles algebraically using Hermitian matrices. This algebraic method allows us to define triangle centers in terms of complex coordinates and matrix equations. For each center, we derive clear formulas that describe its location within the disk and explain the geometric significance behind these expressions.

A major focus of the thesis is understanding when the hyperbolic Euler line exists, i.e., when the circumcenter and centroid lie on a common hyperbolic line, and whether the orthocenter lies on it as well. We identify conditions under which these centers exist, depending on the shape and angle of the triangle. The concepts of circumparallelism and orthoparallelism are introduced to explain when the perpendicular bisectors and altitudes of the triangle become mutually parallel and cease to intersect within the disk. These concepts are unique to hyperbolic geometry and help explain why certain triangles may lack one or more centers.

Throughout the thesis, the focus is on presenting the material in a clear and beginner-friendly way, while still including all the important definitions, formulas, and logical steps. By doing so, the thesis aims to provide an accessible entry point to a rich area of geometry, guided by the results of Strzheletska's original research.

To reach our conclusions, we first establish Hermitian representations for the three key triangle centers in the hyperbolic setting. We then derive the matrix representation of the hyperbolic Euler line and examine whether and when it contains the orthocenter. By analyzing the geometry and algebraic structure of pencils of circles, we classify the conditions under which each center exists. Finally, we prove that if both the circumcenter and the orthocenter of a hyperbolic triangle exist, then the Euler line passes through the orthocenter if and only if the triangle is isosceles. This result shows a clear and interesting link between the symmetry of a triangle and how its centers line up in hyperbolic geometry.

Dearta Zeqiri
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