Algebra 2 **Final exam sheet** **2025**

1. Definition of the field, skew-field. The characteristic, characterization of the characteristic. The prime field. Characterization of the prime field with proof. Subfield, field extension. The degree of the field extensions, the degree theorem. Connections among the finite extension, simple extension and algebraic extension.

2. Polynomials over fields: basic operations, roots of a polynomial, the quotient-remainder theorem. Algebraic and transcendental elements. Minimal polynomial and its properties. Algebraic numbers, algebraic integers. Algebraic closure, algebraically closed fields.

3. Finite fields. Primitive elements, Fermat's little theorem. Number of elements of finite fields. Existence and uniqueness of finite fields. Subfields of a finite field. Frobenius automorphism, automorphisms of finite fields. Construction of finite fields.

4. Splitting field. Criterion to test if a polynomial has multiple roots. Normal extensions. Characterization of normality in terms of splitting field. Separable extension. Galois extension, Galois group. Fundamental theorem of Galois theory. Galois group of a finite field.

5. Cyclotomic extension, cyclotomic polynomial. Theorem about the construction of regular n-gon. Theorem about the construction by ruler and compass. Examples: angle trisection, duplicate the cube and squaring the circle. Normal, subnormal and composition series. Theorem of Jordan and Hölder. Solvable groups. Solvability by radicals. Criterion for the solvability by radicals.

6. Modules. Definition of the modules and algebras. Basic properties, examples. Submodules. Morphisms. Diagrams, exact sequences. The four lemma and the five lemma. First, second and third isomorphism theorems for modules.

7. Tensor product of modules, examples. The universal property. The properties of the tensor product. Tensor product and exactness.

8. Definition of the representation, examples. Group algebra. Irreducible representations, the theorem of Maschke. Schur-lemma. Artin-Wedderburn theorem.