Exercises

1. Show that composition of paths satisfies the following cancellation property: If $f_0 \cdot g_0 \simeq f_1 \cdot g_1$ and $g_0 \simeq g_1$ then $f_0 \simeq f_1$.

2. Show that the change-of-basepoint homomorphism β_h depends only on the homotopy class of h.

3. For a path-connected space X, show that $\pi_1(X)$ is abelian iff all basepoint-change homomorphisms β_h depend only on the endpoints of the path h.

4. A subspace $X \subset \mathbb{R}^n$ is said to be *star-shaped* if there is a point $x_0 \in X$ such that, for each $x \in X$, the line segment from x_0 to x lies in X. Show that if a subspace $X \subset \mathbb{R}^n$ is locally star-shaped, in the sense that every point of X has a star-shaped neighborhood in X, then every path in X is homotopic in X to a piecewise linear path, that is, a path consisting of a finite number of straight line segments traversed at constant speed. Show this applies in particular when X is open or when X is a union of finitely many closed convex sets.

5. Show that for a space X, the following three conditions are equivalent:
(a) Every map $S^1 \to X$ is homotopic to a constant map, with image a point.
(b) Every map $S^1 \to X$ extends to a map $D^2 \to X$.
(c) $\pi_1(X, x_0) = 0$ for all $x_0 \in X$.

Deduce that a space X is simply-connected iff all maps $S^1 \to X$ are homotopic. [In this problem, 'homotopic' means 'homotopic without regard to basepoints.']

6. We can regard $\pi_1(X, x_0)$ as the set of basepoint-preserving homotopy classes of maps $(S^1, s_0) \to (X, x_0)$. Let $[S^1, X]$ be the set of homotopy classes of maps $S^1 \to X$, with no conditions on basepoints. Thus there is a natural map $\Phi : \pi_1(X, x_0) \to [S^1, X]$ obtained by ignoring basepoints. Show that Φ is onto if X is path-connected, and that $\Phi([f]) = \Phi([g])$ iff $[f]$ and $[g]$ are conjugate in $\pi_1(X, x_0)$. Hence Φ induces a one-to-one correspondence between $[S^1, X]$ and the set of conjugacy classes in $\pi_1(X)$, when X is path-connected.

7. Define $f : S^1 \times I \to S^1 \times I$ by $f(\theta, s) = (\theta + 2\pi s, s)$, so f restricts to the identity on the two boundary circles of $S^1 \times I$. Show that f is homotopic to the identity by a homotopy f_t that is stationary on one of the boundary circles, but not by any homotopy f_t that is stationary on both boundary circles. [Consider what f does to the path $s \mapsto (\theta_0, s)$ for fixed $\theta_0 \in S^1$.]

8. Does the Borsuk–Ulam theorem hold for the torus? In other words, for every map $f : S^1 \times S^1 \to \mathbb{R}^2$ must there exist $(x, y) \in S^1 \times S^1$ such that $f(x, y) = f(-x, -y)$?

9. Let A_1, A_2, A_3 be compact sets in $\mathbb{R}^3$. Use the Borsuk–Ulam theorem to show that there is one plane $P \subset \mathbb{R}^3$ that simultaneously divides each A_i into two pieces of equal measure.

10. From the isomorphism $\pi_1(X \times Y, (x_0, y_0)) \approx \pi_1(X, x_0) \times \pi_1(Y, y_0)$ it follows that loops in $X \times \{y_0\}$ and $\{x_0\} \times Y$ represent commuting elements of $\pi_1(X \times Y, (x_0, y_0))$. Construct an explicit homotopy demonstrating this.

11. If X_0 is the path-component of a space X containing the basepoint x_0, show that the inclusion $X_0 \hookrightarrow X$ induces an isomorphism $\pi_1(X_0, x_0) \to \pi_1(X, x_0)$.

12. Show that every homomorphism $\pi_1(S^1) \to \pi_1(S^1)$ can be realized as the induced homomorphism φ_* of a map $\varphi : S^1 \to S^1$.

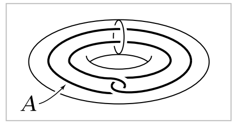
13. Given a space X and a path-connected subspace A containing the basepoint x_0, show that the map $\pi_1(A, x_0) \to \pi_1(X, x_0)$ induced by the inclusion $A \hookrightarrow X$ is surjective iff every path in X with endpoints in A is homotopic to a path in A.

14. Show that the isomorphism $\pi_1(X \times Y) \approx \pi_1(X) \times \pi_1(Y)$ in Proposition 1.12 is given by $[f] \mapsto (p_{1*}([f]), p_{2*}([f]))$ where p_1 and p_2 are the projections of $X \times Y$ onto its two factors.

15. Given a map $f : X \to Y$ and a path $h : I \to X$ from x_0 to x_1, show that $f_* \beta_h = \beta_{fh} f_*$ in the diagram at the right.

$$\begin{array}{ccc} \pi_1(X, x_1) & \xrightarrow{\ \beta_h\ } & \pi_1(X, x_0) \\ \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_*} \\ \pi_1(Y, f(x_1)) & \xrightarrow{\ \beta_{fh}\ } & \pi_1(Y, f(x_0)) \end{array}$$

16. Show that there are no retractions $r : X \to A$ in the following cases:
(a) $X = \mathbb{R}^3$ with A any subspace homeomorphic to S^1.
(b) $X = S^1 \times D^2$ with A its boundary torus $S^1 \times S^1$.
(c) $X = S^1 \times D^2$ and A the circle shown in the figure.
(d) $X = D^2 \vee D^2$ with A its boundary $S^1 \vee S^1$.
(e) X a disk with two points on its boundary identified and A its boundary $S^1 \vee S^1$.
(f) X the Möbius band and A its boundary circle.



17. Construct infinitely many nonhomotopic retractions $S^1 \vee S^1 \to S^1$.

18. Using the technique in the proof of Proposition 1.14, show that if a space X is obtained from a path-connected subspace A by attaching a cell e^n with $n \geq 2$, then the inclusion $A \hookrightarrow X$ induces a surjection on π_1. Apply this to show:
(a) The wedge sum $S^1 \vee S^2$ has fundamental group $\mathbb{Z}$.
(b) For a path-connected CW complex X the inclusion map $X^1 \hookrightarrow X$ of its 1-skeleton induces a surjection $\pi_1(X^1) \to \pi_1(X)$. [For the case that X has infinitely many cells, see Proposition A.1 in the Appendix.]

19. Modify the proof of Proposition 1.14 to show that if X is a path-connected 1-dimensional CW complex with basepoint x_0 a 0-cell, then every loop in X is homotopic to a loop consisting of a finite sequence of edges traversed monotonically. [This gives an elementary proof that $\pi_1(S^1)$ is cyclic, generated by the standard loop winding once around the circle. The more difficult part of the calculation of $\pi_1(S^1)$ is therefore the fact that no iterate of this loop is nullhomotopic.]