Konvex geometria
Matematikus szakos hallgatóknak

Gyakorló feladatok a záróvizsgához

Exercise 1. Let $S \subseteq \mathbb{R}^n$ be an arbitrary set. Let the *kernel* of S be the set of points $x \in S$ for which $[x, y] \subseteq S$ for any $y \in S$. Prove that the kernel of S is convex.

Exercise 2. Consider an $(n \times n)$ matrix A as a point of the space $\mathbb{R}^{n^2}$. Denote by $\mathcal{S}$, $\mathcal{S}_+$ and $\mathcal{S}_{++}$ the families of symmetric. positive definit and positive semidefinite $(n \times n)$ matrices in $\mathbb{R}^{n^2}$. Prove that these sets are convex.

Exercise 3. Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be an invertible linear transformation.

(a) Prove that the set $T(K) = \{Y : \exists X \in K, T(x) = y\}$ is convex for any convex set $K \subseteq \mathbb{R}^n$.

(b) Prove that the sets

$$P = \{Y \in \mathbb{R}^n : \langle y, x_i \rangle \leq \alpha_i, i = 1, 2, 3 \ldots, k\}$$

and $T(P)$ are convex, and that there are vectors $w_1, w_2, \ldots, w_k \in \mathbb{R}^n$ and scalars $\beta_1, \beta_2, \ldots, \beta_k \in \mathbb{R}$ such that

$$T(P) = \{Y \in \mathbb{R}^n : \langle y, w_i \rangle \leq \beta_k, i = 1, 2, 2 \ldots, k\}.$$

Exercise 4. Show that

(a) the finite Helly theorem is false for nonconvex sets in $\mathbb{R}^n$, or if we assume that the intersection of every n elements of the family is not empty,

(b) there is a family of infinitely many closed, convex sets in $\mathbb{R}^n$ in which the intersection of every at most $n+1$ elements is nonempty, but the intersection of all elements is empty.

Exercise 5. Let $\mathcal{F}$ be a finite family of convex sets in $\mathbb{R}^n$, and $C \subset \mathbb{R}^n$ be convex. Prove that if for any at most $n+1$ elements of $\mathcal{F}$ there is a translate of C that intersects/contains/is contained in each element, then there is a translate of C that intersects/contains/is contained, respectively in each element of C.

Exercise 6. Let $L, L' \subset \mathbb{R}^n$ be linear subspaces of dimensions k, $n-k$, respectively, for some $0 \leq k \leq n$, where $L \cap L' = \{o\}$.

a) Prove that for every $p, q \in \mathbb{R}^n$, the intersection of the affine subspaces $p + L$ and $q + L'$ is a singleton.

b) Let $F = p + L$ be fixed. Define the function $\pi : \mathbb{E}^n \to F$ in the following way: $f(q) = q'$, if the intersection of F and $q + L'$ is $\{q'\}$ (the name of the map π is *projection onto F parallel to L'*). Prove that for any convex set $K \subseteq \mathbb{R}^n$, $\pi(K)$ is convex, and for any convx set $K' \subseteq F$, $\pi^{-1}(K')$ is convex.

Exercise 7. Let K and L be plane convex bodies. Prove that then $\mathrm{perim}(K + L) = \mathrm{perim}(K) + \mathrm{perim}(L)$.

Exercise 8. Prove that if $K \subseteq \mathbb{R}^n$ is closed, convex and unbounded, then for every point $p \in K$, K contains a closed half line starting at p.

Exercise 9. Let $K \subset \mathbb{R}^n$ be a closed, convex set, and let F be a proper face of K. Prove that if p is an extremal face of F, then p is an extremal face of Ka as well.

Exercise 10. Let $A \subset \mathbb{R}^n$ be arbitrary. Prove that p is an extremal point of $\operatorname{conv}(A)$ if and only if $p \notin \operatorname{conv}(A \setminus \{p\}$.

Exercise 11. Let $K_1, K_2, K_3 \subset \mathbb{R}^n$ be closed, convex sets whose union is also convex. Prove that if the intersection of any two of them is nonepmty, then the intersection of all three of them is nonempty.

Exercise 12. Prove that there is a valuation on $\mathcal{K}(\mathbb{R}^n)$ whose value on the indicator function of compact, convex set K is the volume of K.

Exercise 13. Prove that the exposed points of a compact, convex set are also extremal points of the set, and the extremal points of a polytope are also exposed points of the polytope.

Exercise 14. Prove that every n-dimensional polytope has a facet. Prove that for every $k = 0, 1, \ldots, n$, every n-dimensional polytope has a k-dimensional face.

Exercise 15. Let P be an arbitrary n-dimensional polytope, and let $F \subset G$ be proper faces of P such that $\dim F + 2 = \dim G$. Prove that P has exactly two faces F' satisfying $F \subsetneq F' \subsetneq G$.

Exercise 16. Let $P \subset \mathbb{R}^n$ be an n-dimensional polytope. Let H be a hyperplane passing through an interior point of P and contains no vertex of P. Let H^+ be one of the two open half spaces bounded by H, and let f_i^+ P be the number of the i-dimensional faces of P contained in H^+. Then

$$\sum_{i=1}^{n-1} (-1)^i f_i^+ = 1.$$

Exercise 17. Let $P \subset \mathbb{R}^n$ be an n-dimensional polytope, and let F be a k-dimensional face of P. Denote by $f_j(F, P)$ the number of those j-dimensional faces of P that contain F. Prove that

$$\sum_{j=k}^{n-1} (-1)^j f_j(F, P) = (-1)^{n-1}.$$

Exercise 18. Prove that for any nonenpty set $A \subseteq \mathbb{R}^n$, $((A^\circ)^\circ)^\circ = A^\circ$.

Exercise 19. Let $A \subseteq \mathbb{R}^n$ be nonempty. Show that $(A^\circ)^\circ$ is the closure of the set $\operatorname{conv}(A \cup \{o\})$.

Exercise 20. Let $A \subseteq \mathbb{R}^n$ be a nonempty set satisfying $A^\circ = A$. Show that A is the closed unit ball centered at o.

Exercise 21. Let P be an n-dimensional polytope, and let $\lceil \frac{n}{2} \rceil < k < n+1$ be an integer. Prove that if the convex hull of any k vertices of P is a face of P, then P is a simplex.

Exercise 22. Prove that the support function of the cube $K = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : -1 \leq x_i \leq 1, i = 1, 2, \ldots, n\}$ is $h((u_1, \ldots, u_n)) = \sum_{i=1}^n |u_i|, (u_1, \ldots, u_n) \in \mathbb{R}^n$.

Exercise 23. Let K be a convex body in $\mathbb{R}^n$. Prove that then there is some point $x \in \mathbb{R}^n$ and a simplex T such that

$$x + T \subseteq K \subseteq x - nT.$$

Exercise 24. Based on the previous problem, show that the Banach-Mazur distance of any two convex bodies is at most n^4.