Probability Theory 2

List of exercises of final exam

FE.1 Let X and Y be independent random variables with distribution $EXP(\lambda)$ and $EXP(\mu)$. Find the density of $Z := X + Y$.

FE.2 Find the density function of the sum of n i.i.d. random variables with distribution $EXP(\lambda)$.

FE.3 May B. Dunn is a student in mathematics on BUTE. She tries to pass the Probability Theory 2 course. First, she needs to get the signature in the practical part. If she fails in one semester, she tries again in the next one. The semesters are independent and in each of the semesters the probability that she gets the signature is $1/3$. If she gets the signature she will try the oral exam on the theory. Again, if she fails she tries in the next semester, the semesters are independent and in each of the semesters, the probability that she passes the oral exam is $1/4$. Find the distribution of the number of semesters required for May B. Dunn to pass.

FE.4 We roll a die until we see two consecutive 6. Denote ν the number of required rolls. Determine the prob. generating functon of ν, and using this, the expected value and the variance.

FE.5 Let $\xi_1, \xi_2, \dots$ be i.i.d random variables with distribution: $\mathbb{P}\big(\xi_i = 1\big) = p$, $\mathbb{P}\big(\xi_i = 0\big) = 1 - p$, where $0 < p < 1$. Let

$$\nu_{\alpha\beta} := \min\{n \geq 2 : \xi_{n-1} = \alpha,\ \xi_n = \beta\}, \qquad \alpha, \beta \in \{0, 1\}.$$

Determine the prob. gen. function of $\nu_{\alpha\beta}$, and using this, the expected value and variance, for every possible combination of α, β.

FE.6 Let $X_1, X_2, X_3, \dots$ be a sequence of independent and identically distributed random variables, with distribution function $F(x) := \mathbb{P}\big(X_i < x\big)$. Let ν be an $\mathbb{N}$-valued random variable independent of X_i's, and denote $G(z)$ the probability generating function of ν. Show that the distribution function of $Y := \max\{X_1, X_2, \dots, X_\nu\}$ is $H(x) = G(F(x))$.

FE.7 Let U be a random variable with distribution $UNI(0, 1)$, and let X be the random variable, which conditional distribution is $BIN(n, U)$ conditioned on U. Prove that the distribution of X is $UNI\{0, 1, \dots, n\}$.

FE.8 Let Λ be a random variable with distribution $EXP(\mu)$ and let X be the random variable, which conditional distribution is $POI(\Lambda)$ conditioned on Λ. Determine the distribution of X.

FE.9 Let $X_1, X_2, \dots$ be i.i.d. ($\mathbb{N}$ valued) random variables and let ν be a random variable independent of X_i's. Let $Y = \sum_{k=1}^{\nu} X_k$. Show that

$$\mathbb{D}^2(Y) = \mathbb{D}^2(\nu)\mathbb{E}(X_1)^2 + \mathbb{E}(\nu)\mathbb{D}^2(X_1).$$

FE.10 Let us consider a branching process, for which the probability generating function of the offsprings is $P(z)$. Denote X the size of the whole population (i.e. the number of all individuals who ever lived). Denote $Q(z) = \mathbb{E}(z^X)$. Prove that $Q(z)$ is the inverse of $z/P(z)$!

FE.11 Consider a branching process for which the expected value of the offsprings is 1 and the variance is $0 < \sigma < 1$. Denote X_n the number of individuals in the nth generation. That is, $X_0 = 1$, X_1 is the number of children of the first individual, etc.

(a) What is the expected value of the size of the population?
(b) Give a formula for $\mathbb{D}^2(X_n)$! Prove by induction!

FE.12 Let $X_1, \dots, X_9$ be independent random variables with distribution $UNI[0,1]$. Moreover, let $Y = \sqrt[9]{X_1 \cdots X_9}$. Using Chebisev's inequality, give a lower estimate for the probability

$$\mathbb{P}(e^{-5/3} < Y < e^{-1/3})!$$

FE.13 Let $X_1, X_2, \dots$ be uncorrelated random variables with finite variance, 0 expected value. (That is, for every $i \geq 1$, $\mathbb{E}(X_i) = 0$, $\sigma_i^2 := \mathbb{D}^2(X_i) = \mathbb{E}(X_i^2) < \infty$, and for every $i \neq j$, $\mathbb{E}(X_i X_j) = 0$). Let $S_n := X_1 + X_2 + \cdots + X_n$. Show that if $\lim_{i\to\infty} \sigma_i^2 / i = 0$ then $\lim_{n\to\infty} \mathbb{P}(|S_n/n| > \delta) = 0$ for every $\delta > 0$.

FE.14 (a) Let X be a random variable. We call the function $R(t) = \mathbb{E}(e^{tX})$ the moment generating function of X. Show that for every $x \in \mathbb{R}$, $\mathbb{P}(X > x) \leq \inf_{t>0} R(t) e^{-tx}$.
(b) Show that the function $t \mapsto \log R(t)$ is convex.
(c) Let X be a random variable with distribution $POI(\lambda)$. Using the exercise **FE.14a**, estimate $\mathbb{P}(X > x)$.

FE.15 We toss a coin 60 times and denote the number of heads by X. Give an upper bound for the probability

$$\mathbb{P}(|X - 30| \geq 20)$$

by using Chebisev's inequality. A better estimate can be given by using the turbo-Markov inequality:
(a) Let $Y_\beta = e^{\beta X}$, where $0 < \beta$. Show that $\mathbb{E}(Y_\beta) = 2^{-60}(1 + e^\beta)^{60}$.
(b) Give an upper estimate for $\mathbb{P}(X \geq 50)$ by using Markov-inequality for the non-negative random variable Y_β for all $\beta > 0$.
(c) Find the optimal β, that is, find the minimum of the estimate in (b). (This can be done by minimizing the convex function $f(\beta) = \log(1 + e^\beta) - \frac{5}{6}\beta$.)
(d) Combining the previous points, show $\mathbb{P}(|X - 30| \geq 20) \leq 2 \cdot 3^{60} \cdot 5^{-50} < 10^{-6}$.

FE.16 Let Y_n be a sequence of bounded random variable (i.e. $\mathbb{P}(|Y_n| < M) = 1$ for some $M > 0$ and every n) and suppose that $Y_n \xrightarrow{\mathbb{P}} Y$ (i.e. $\mathbb{P}(|Y_n - Y| > \delta) \to 0$ for every $\delta > 0$). Then $\mathbb{E}(|Y_n - Y|) \to 0$ (i.e. $Y_n \xrightarrow{L^1} Y$) as $n \to \infty$, and in particular $\mathbb{E}(Y_n) \to \mathbb{E}(Y)$.

FE.17 Let X_n be a sequence of random variables and let Y, X be a random variables such that $\mathbb{P}(|X_n| \leq Y) = 1$, $X_n \xrightarrow{\text{a.s.}} X$ and $\mathbb{E}(|Y|) < \infty$. Then $X_n \xrightarrow{L^1} X$.
(Hint: Apply Fatou's Lemma for $2Y - |X_n - X|$.)

FE.18 Let $X_1, X_2, \dots$ and Y be random variables on the same prob. space $(\Omega, \mathcal{A}, \mathbb{P})$ and let $F_1(x)$, $F_2(x)$, ... and $G(x)$ be their distribution functions respectively. Show that if $X_n \xrightarrow{\mathbb{P}} Y$ then $\lim_{n\to\infty} F_n(x) = G(x)$ at every continuity point x of G.

HWFE.19 Let $f : [0,1] \to \mathbb{R}$ be continuous. Show that

$$\lim_{n\to\infty} \int_0^1 \int_0^1 \cdots \int_0^1 f\left(\frac{x_1 + x_2 + \cdots + x_n}{n}\right) dx_1\, dx_2 \cdots dx_n = f\left(\frac{1}{2}\right),$$

$$\lim_{n\to\infty} \int_0^1 \int_0^1 \cdots \int_0^1 f\left((x_1 x_2 \cdots x_n)^{1/n}\right) dx_1\, dx_2 \cdots dx_n = f\left(\frac{1}{e}\right).$$

FE.20 Show that
$$\lim_{n\to\infty}\int_0^1\int_0^1\cdots\int_0^1 \frac{x_1^2+x_2^2+\cdots+x_n^2}{x_1+x_2+\cdots+x_n}dx_1\,dx_2\,\cdots\,dx_n=\frac{2}{3}.$$

FE.21 Let $X_1, X_2, \ldots$ be independent random variables such that
$$\mathbb{P}\Big(X_n=n^2-1\Big)=n^{-2},\quad \mathbb{P}\Big(X_n=-1\Big)=1-n^{-2}.$$
Prove that for every $n\in\mathbb{N}$, $\mathbb{E}\big(X_n\big)=0$ but
$$\lim_{n\to\infty}\frac{X_1+X_2+\cdots+X_n}{n}=-1\qquad\text{almost surely.}$$

FE.22 Let X_n be i.i.d random variables with $X_n\sim GEO(p)$. That is $\mathbb{P}(X_n=k)=(1-p)p^k$ for $k\geq 0$. Show that $\limsup_{n\to\infty}\frac{X_n}{\log n}=|\log p|^{-1}$ almost surely.

FE.23 Let $X_1, X_2, \ldots$ be independent random variables such that $\mathbb{P}(X_n=1)=p_n$ and $\mathbb{P}(X_n=0)=1-p_n$. Which properties does $p_n, n=1,2,\ldots$ have if
(a) $X_n\xrightarrow{\mathbb{P}}0$ as $n\to\infty$
(b) $X_n\xrightarrow{\text{a.s.}}0$ as $n\to\infty$.

FE.24 We make infinitely many independent experiments. The probability that the nth experiment is successful is $n^{-\alpha}$, where $0<\alpha<1$. Let $k\geq 1$. It makes us happy if it happens infinitely often that we have k consecutive successful experiments. What is the probability that we are happy?

FE.25 Let X be a random variable with standard normal distribution $N(0,1)$. We have shown earlier that for any $x>0$
$$\left(\frac{1}{x}-\frac{1}{x^3}\right)\frac{\exp(-x^2/2)}{\sqrt{2\pi}}\leq\mathbb{P}\Big(X>x\Big)\leq\frac{1}{x}\frac{\exp(-x^2/2)}{\sqrt{2\pi}}.$$
(a) Let $X_1, X_2, \ldots$ be i.i.d with $X_i\sim N(0,1)$. Show that
$$\mathbb{P}\Big(\limsup_{n\to\infty}\frac{X_n}{\sqrt{2\log n}}=1\Big)=1.$$
(b) Let $S_n:=X_1+X_2+\cdots+X_n$, where $X_1, X_2, \ldots$ are from (a). Show that for every $C>\sqrt{2}$
$$\mathbb{P}\Big(\limsup_{n\to\infty}\frac{S_n}{\sqrt{n\log n}}<C\Big)=1.$$

FE.26 (a) Show that $\phi_{aX+b}(t)=e^{itb}\varphi_X(at)$.
(b) Show if X, Y are independent the characteristic function ϕ_{X+Y} of $X+Y$ is the product of the characteristic functions ϕ_X, ϕ_Y.
(c) Show an example that $\phi_{X+Y}=\phi_X\phi_Y$ but X and Y are not independent.

FE.27 (a) Let ϕ be a characteristic function. Show that $\overline{\phi}, \phi^2, |\phi|^2$ are characteristic functions too.

(b) Let $\phi_1,\ldots,\phi_n$ be characteristic functions. Show that $\sum_{i=1}^n q_i\phi_i$ is a characteristic function for any $q_1,\ldots,q_n\geq 0$ with $\sum_{i=1}^n q_i=1$. In fact, show that $\mathrm{Re}(\phi)$ is a characteristic function too.

FE.28 Let X be a random variable and let ϕ_X the characteristic function of X. Suppose that there exists a $t_0\in\mathbb{R}$ such that $t_0\neq 0$ and $|\phi_X(t_0)|=1$. Show that X has lattice distribution. That is, there exists an arithmetic sequence $p_n=an+b$ with some $a\neq 0, b\in\mathbb{R}$ such that $\mathbb{P}(X\in\{p_n\}_{n\in\mathbb{Z}})=1$.

FE.29 Let U be a random variable with distribution $UNI[0,1]$, and let Y and Z be independent and independent of U with distribution $EXP(1)$. Show that $X = U(Y+Z)$ has distribution $EXP(1)$ as well.

FE.30 Let X and Y be i.i.d random variables with expected value 0 and variance 1. Denote the common characteristic function by φ. Suppose that $X+Y$ and $X-Y$ are independent. Show that this is possible only if X and Y are standard normal random variables.

(*Hint:* Show that in this case $\varphi(2t) = \varphi(t)^2|\varphi(t)|^2$, and find the limit of $\varphi(t/2^n)^{2^n}|\varphi(t/2^n)|^{4^n-2^n}$.)

FE.31 Let $S = [0,1]$ and let μ_n be the discrete measure such that every point k/n has weight $1/(n+1)$ for $k = 0, 1, \dots, n$. Show that $\mu_n \Rightarrow \mu$, where μ is the Lebesgue measure on S. (Use the definition of the weak convergence, not the equivalent formalisations for distribution functions.)

FE.32 Let $X_1, X_2, \dots$ be i.i.d random variables with distribution function $F(x) := \mathbb{P}\big(X_i < x\big)$. Let $M_n := \max\{X_1, X_2, \dots, X_n\}$. Suppose that $F(x) < 1$ for every $x < \infty$ and $\lim_{x\to\infty} x^\alpha\big(1-F(x)\big) = b$ for some $\alpha, b \in (0,\infty)$ (that is: $1 - F(x) \sim x^{-\alpha}$ as $x \to \infty$). Show that the distribution of $n^{-1/\alpha}M_n$ converges weakly to:

$$\mathbb{P}\left(n^{-1/\alpha}M_n < x\right) \to \mathbb{1}_{\{x>0\}} \exp\left(-bx^{-\alpha}\right).$$

FE.33 Let $X_n \sim POI(n)$ be independent random variables. Show that

$$\frac{X_n - n}{\sqrt{X_n}} \xrightarrow{\mathbf{D}} N(0,1) \text{ as } n, m \to \infty.$$

FE.34 Show with the method of characteristic functions that $\mathrm{BIN}(n, p_n) \xrightarrow{\mathbf{D}} \mathrm{POI}(\lambda)$ as $n \to \infty$ if $\lim_{n\to\infty} np_n = \lambda$.

FE.35 Prove the weak law of large numbers by using the method of characteristic functions.