Tools of Modern Probability
sample exercises for the final exam, June 2021

1. Let c_n denote the ($n-1$-dimensional) surface volume of the $n-1$-dimensional unit sphere $S_n := \{x \in \mathbb{R}^n : |x| = 1\}$. Let d_n denote the (n-dimensional) volume of the n-dimensional unit ball $B_n := \{x \in \mathbb{R}^n : |x| \le 1\}$. Give and prove the relation between c_n and d_n.

2. Describe the asymptotic behaviour of the sequence $I_n = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^n x \, \mathrm{d}x$.

3. Calculate $\int_0^\infty x^\alpha e^{-x} \, \mathrm{d}x$ for nonnegative integer and half-integer values of α.

4. Describe the asymptotic behaviour of the sequence $I_n = \frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)\sqrt{n}}$ as $n \to \infty$, where Γ is the Euler gamma function.

5. Let χ be counting measure on $\mathbb{Z}$. Calculate $\int_{\mathbb{Z}} e^{-|x|} \, \mathrm{d}\chi(x)$.

6. Give an example of a function $f : [0,1] \times [0,1] \to \mathbb{R}$ for which the integrals

$$I_1 := \int_0^1 \int_0^1 f(x,y) \, \mathrm{d}y \, \mathrm{d}x$$
$$I_2 := \int_0^1 \int_0^1 f(x,y) \, \mathrm{d}x \, \mathrm{d}y$$

both make sense, but they are not equal, *or* show that there is no such function.

7. Let μ be a measure on $\mathbb{R}$ which has density f w.r.t. Lebesgue measure, where

$$f(x) = \begin{cases} x+1 & \text{if } 0 < x < 1 \\ 0 & \text{if not} \end{cases}.$$

Let $g(x) = \sqrt{x}$ and let $\nu = g_* \mu$ be the push-forward of μ with g.

a.) Calculate $\nu([\frac{1}{2}, \frac{2}{3}])$.

b.) Show that ν is also absolutely continuous w.r.t. Lebesgue measure and calculate the density.

8. Let μ be the uniform distribution on $[-1,1]$. Let $f(x) = 2x$ and let $\nu = f_* \mu$. Calculate the characteristic function of ν.

9. Let (X,Y) be uniformly distributed on the square $[-1,1]^2$. Let $Z = X + XY$, and let μ be the distribution of the random variable μ. Calculate $\int_{\mathbb{R}} z^2 \, \mathrm{d}\mu(z)$.

10. Is the following statement true? If yes, prove it. I no, give a counterexample:

Let $X_1, X_1, \ldots$ and X_∞ be real-valued random variables, which are all absolutely continuous w.r.t. Lebesgue measure, with densities $f_1, f_2, \ldots$ and f_∞, respectively. Now if X_n converges to X_∞ weakly, then $f_n(x) \to f_\infty(x)$ for Lebesgue almost every $x \in \mathbb{R}$.

11. We roll two fair dice. Let X and Y be the two numbers rolled. Calculate $\mathbb{E}(X+Y|X-Y)$.

12. Let the random point (U,V) be uniformly distributed on the triangle with corners $(0,0)$; $(1,0)$; $(1,0)$ and let $Z = \mathbf{1}_{\{V > \frac{1}{2}\}}$. Calculate $\mathbb{E}(Z|U)$.

13. Let X be uniformly distributed on $[-1,2]$. Calculate $\mathbb{E}(X|X^2)$.