THE ČERNÝ CONJECTURE FOR AUTOMATA WITH BLOCKING STATES

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Dedicated to Professor Ferenc Gécseg on his 70th birthday

Abstract

In [7], Jan Černý conjectured that an arbitrary directable automaton with n states has a directing word of length not longer than $(n-1)^2$. This conjecture is one of the most longstanding open problems in the theory of finite automata. Most of papers dealing with this conjecture reduce the problem to special classes of automata. In present paper we deal with this conjecture in the class of automata having a blocking state. We prove that the conjecture is true in this class of automata. We show that if an automaton has n states and contains a blocking state then it has a directing word whose length is not longer than $\frac{n(n-1)}{2}$. The notion of the blocking state is a generalization of the notion of the trap for directable automata. Thus every trap-directable automaton with n states has a directing word of length not longer than $\frac{n(n-1)}{2}$. We give an example for trap-directable automaton with n states in which the length of the shortest directing word is $\frac{n(n-1)}{2}$. We prove that if the Černý Conjecture holds for a subautomaton of a directable automaton then it holds for the automaton.

Keywords: directing word and directed state of automata, Černý Conjecture

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1 Preliminaries

By an *automaton* $\mathbf{A} = (A, X, \delta)$ we mean always a deterministic automaton without outputs, where $A \neq \emptyset$ is the *state set*, $X \neq \emptyset$ is the *input set* and $\delta : A \times X \to A$ is the *transition function*. We denote the cardinality of the set A with $|A|$. The automaton $\mathbf{A}$ is *A-finite* if $|A| < \infty$.

The *input monoid* [*semigroup*] X^* [X^+] of $\mathbf{A}$ is the free monoid [free semigroup] over X. The empty word is denoted by e. Let $|p|$ be *the length of the input word* $p \in X^*$. More precisely, $|e| = 0$ and if $p = x_1x_2 \ldots x_k$ $(x_1, x_2, \ldots, x_k \in X)$ then $|p| = k$.

The transition function δ can be extended in the usual way: Let $\delta(a, e) = a$ for every $a \in A$. If $a \in A$, $p \in X^*$ and $x \in X$ then let $\delta(a, px) = \delta(\delta(a, p), x)$. For brevity we shall use the notation ap instead of $\delta(a, p)$.

If $B \subseteq A$ and $M \subseteq X^*$ then let $BM = \{ap;\ a \in B,\ p \in M\}$. If $a \in A$ or $p \in X^*$ then $\{a\}M$ and $B\{p\}$ will be denoted by aM and Bp, respectively.

The *characteristic semigroup* $S(\mathbf{A})$ of the automaton $\mathbf{A}$ is the factor semigroup X^*/ρ_A, where ρ_A defined by $(p, q) \in \rho_A$ if and only if $ap = aq$ for all $a \in A$.

A mapping $\varphi : A \to B$ is a *homomorphism* of the automaton $\mathbf{A} = (A, X, \delta)$ into the automaton $\mathbf{B} = (B, X, \delta')$ if $\varphi(ax) = \varphi(a)x$ for all $a \in A$ and $x \in X$. If φ is a bijective mapping, then φ is an *isomorphism* of $\mathbf{A}$ onto $\mathbf{B}$ and $\mathbf{A}$ *is isomorphic with* $\mathbf{B}$.

The automaton $\mathbf{A}' = (A', X, \delta')$ is a *subautomaton* of the automaton $\mathbf{A} = (A, X, \delta)$ if $A' \subseteq A$ and δ' is the restriction of δ to $A' \times X$. If $A' = \{c\}$ then the state $c \in A$ is called a *trap* of $\mathbf{A}$. If an automaton $\mathbf{A}$ has a trap then we say that the automaton $\mathbf{A}$ is a *trapped automaton*.

The automaton $\mathbf{A} = (A, X, \delta)$ is a (*finite*) *direct product* of of the automata $\mathbf{A}_i = (A_i, X, \delta_i)$ $(i = 1, 2, \ldots, n)$ if $A = A_1 \times A_2 \times \cdots \times A_n$ and for every $a_i \in A_i$ $(i = 1, 2, \ldots, n)$ and $x \in X$

$$\delta((a_1, a_2, \ldots, a_n), x) = (\delta_1(a_1, x), \delta_2(a_2, x), \ldots, \delta(a_n, x)).$$

The automaton $\mathbf{A} = (A, X, \delta)$ is *connected* if, for arbitrary states $a, b \in A$, there exist $p, q \in X^*$ such that $ap = bq$. Specially, $\mathbf{A}$ is *strongly connected* if $q = e$, that is, for arbitrary states $a, b \in A$, there exists a $p \in X^*$ such that $ap = b$. The automaton $\mathbf{A}$ is *strongly trap-connected* if it has a trap c and for every state $a \in A - \{c\}$ and $b \in A$, there exists $p \in X^*$ such that $ap = b$. It is known that the automaton $\mathbf{A}$ is strongly connected if and only if it has no subautomaton $\mathbf{A}' = (A', X, \delta')$ of $\mathbf{A}$

such that $A' \neq A$. Furthermore, if $\mathbf{A}$ is strongly trap-connected then it has only one trap.

2 Directable automata

An automaton $\mathbf{A}$ is *directable* or *synchronizable* if there exist $p \in X^*$ such that $|Ap| = 1$, that is, $Ap = d$ for some state $d \in A$. The input word p and the state d is called *a directing word* and *a directed state* of the automaton $\mathbf{A}$, respectively. Every directable automaton is a connected automaton. A directable automaton $\mathbf{A}$ is *trap-directable* if it has a trap. If an automaton $\mathbf{A}$ is trap-directable then it is trap-connected and the trap is the only directed state. Evidently, every homomorphic image and every subautomaton of a directable automaton is also directable. Furthermore finite direct product of directable automata is directable. Let $\mathbf{A} = (A, X, \delta)$ be a directable automata with n states. If $D(\mathbf{A})$ denotes the set of directing words of $\mathbf{A}$ then let

$$d(\mathbf{A}) = \min\{|p|;\ p \in D(\mathbf{A})\}.$$

Denote $\mathcal{D}_n$ the class of directable automata with n states and

$$d(n) = \max\{d(\mathbf{A});\ \mathbf{A} \in \mathcal{D}_n\}.$$

Jan Černy has constructed an n state automaton $\mathbf{A}$ for which $d(\mathbf{A}) = (n-1)^2$ ([7]). This means that $(n-1)^2 \leq d(n)$. In a number of special classes of automata it is prowed that $d(n) \leq (n-1)^2$ ([1], [2], [3], [12]).

Černý Conjecture: For every positive integer n, $d(n) = (n-1)^2$.

For an arbitrary (not empty) subclass $\mathcal{K}_n$ of the class $\mathcal{D}_n$ of directable automata with n states, let

$$d_{\mathcal{K}}(n) = \max\{d(\mathbf{A});\ \mathbf{A} \in \mathcal{K}_n\}.$$

If $\mathcal{K}_n$ and $\mathcal{L}_n$ are subclasses of $\mathcal{D}_n$ such that

$$\emptyset \subset \mathcal{K}_n \subset \mathcal{L}_n \subset \mathcal{D}_n,$$

then

$$d_{\mathcal{K}}(n) \leq d_{\mathcal{L}}(n) \leq d(n).$$

If $\mathcal{CD}_n$ is the class of commutative directable automata with n states then $d_{\mathcal{CD}}(n) = n - 1$ ([20], [10]). An automaton is *aperiodic* if its characteristic semigroup has only trivial subgroups. In [24], A. N. Trahtman has proved that if $\mathcal{AD}_n$ is the class of aperiodic directable automata with n states then $d_{\mathcal{AD}}(n) \leq \frac{n(n-1)}{2}$. In [14], T. Petković and M. Steinby introduced a stronger form of directability. The automaton $\mathbf{A} = (A, X, \delta)$ is *piecewise directable* if there exist $x_1, x_2, \ldots, x_k \in X$ such that

$$p \in X^* x_1 X^* x_2 X^* \ldots X^* x_k X^* \quad \Longrightarrow \quad ap = bp$$

for all $a, b \in A$. If $\mathcal{PD}_n$ is the class of the piecewise directable automata with n states then $d_{\mathcal{PD}}(n) \leq \frac{n(n-1)}{2}$ ([14]).

For a general survey of theory of directable automata the reader is referred to [5].

3 Automata with blocking states

Let $\mathbf{A} = (A, X, \delta)$ be an arbitrary automaton. For every $a, b \in A$, consider the sets

$$U(a,b) = \{p \in X^*;\ ap = b\},$$

$$V(a,b) = \{q \in U(a,b);\ |q| \leq |p|,\ \text{for every } p \in U(a,b)\}.$$

It is evident that $V(a,b) \subseteq X^k$ for any integer k with $0 \leq k \leq n-1$. The state $c \in A$ will be called *a blocking state* of the automaton $\mathbf{A}$, if for every $a \in A$,

$$U(a,c) \neq \emptyset \quad \text{and} \quad cV(a,c) = c.$$

If the automaton $\mathbf{A}$ has a blocking state then $\mathbf{A}$ is a connected automaton. Furthermore, if $\mathbf{A}$ is trap-connected, then the trap is the only blocking state of $\mathbf{A}$.

Lemma 1 *([11]) An A-finite automaton $\mathbf{A} = (A, X, \delta)$ is directable if and only if, for every $a, b \in A$, there exists $p \in X^*$ such that $ap = bp$.*

Theorem 2 *If $\mathcal{B}_n$ is the class of n-state automata with blocking states then $\mathcal{B}_n \subset \mathcal{D}_n$ and $d_{\mathcal{B}}(n) = \frac{n(n-1)}{2}$.*

Proof. Let $\mathbf{A} = (A, X, \delta)$ be an n-state automaton with blocking state c. If $a, b \in A$, $p \in V(a,c)$ and $q \in V(bp, c)$ then

$$b(pq) = (bp)q = cq = (cp)q = (ap)q = a(pq).$$

By Lemma 1, the automaton $\mathbf{A}$ is directable. But not every directable automaton has a blocking state, therefore $\mathcal{B}_n \subset \mathcal{D}_n$.

If $n = 1$ then the statement is obvious. Assume $n > 1$. Let $d \in A$ be a state such that, for every $a \in A$, the assumptions $q \in V(a, c)$ and $r \in V(d, c)$ together imply $|q| \leq |r|$. If $p \in V(d, c)$ and

$$p = x_1 x_2 \dots x_{n-k}, \quad x_1, x_2, \dots, x_{n-k} \in X, \quad 1 \leq k \leq n-1,$$

then

$$d,\ dx_1,\ dx_1x_2,\ \dots,\ dx_1x_2\dots x_{n-k-1},\ dx_1x_2\dots x_{n-k-1}x_{n-k} = c$$

are different states. Furthermore, for every $1 \leq i \leq n-k$,

$$x_i x_{i+1} \dots x_{n-k} \in V(dx_1x_2\dots x_{i-1}, c) \quad (x_0 = e).$$

From this it follows that if

$$p = x_{n-k}(x_{n-k-1}x_{n-k})\dots(x_2\dots x_{n-k-1}x_{n-k})(x_1x_2\dots x_{n-k-1}x_{n-k}),$$

then

$$|Ap| \leq k \quad \text{and} \quad c \in Ap.$$

Thus there are $q_1, q_2, \dots, q_{k-1} \in X^*$ such that $|q_1|, |q_2|, \dots, |q_{k-1}| \leq n-k$ and

$$Apq_1q_2\dots q_{k-1} = c,$$

that is,

$$|pq_1q_2\dots q_{k-1}| \leq \frac{(n-k)(n-k+1)}{2} + (k-1)(n-k) =$$

$$= \frac{(n-k)(n-1+k)}{2} = \frac{n(n-1)+k(1-k)}{2} \leq \frac{n(n-1)}{2}.$$

This means that $d_{\mathcal{B}}(n) \leq \frac{n(n-1)}{2}$.

Consider the trap-directable automaton $\mathbf{T} = (T, X, \delta)$ with the state set $T = \{0, 1, \dots, n-1\}$, the input set $X = \{x_1, x_2, \dots, x_{n-1}\}$ $(2 \leq n)$ and the transition function δ defined by

$$\delta(i, x_i) = i-1, \quad \delta(0, x_i) = 0 \quad (i = 1, 2, \dots, n-1),$$

$$\delta(i, x_{i+1}) = i+1 \quad (i = 1, 2, \dots, n-2),$$

$$\delta(i, x_s) = i \quad (i = 1, 2, \dots, n-2,\ s \neq i, i+1),$$

$$\delta(n-1, x_s) = n-1, \quad (s \neq n-1).$$

It is easy to see that, for every integer $1 \leq i \leq n-1$, the shortest word $r \in X^+$ is $r = x_i \dots x_2 x_1$ such that

$$\{0, i, i+1, \dots, n-1\}r = \{0, i+1, \dots, n-1\}.$$

Using this fact, it is a matter of calculating to see that the input word

$$p = x_1(x_2x_1)(x_3x_2x_1)\dots(x_{n-2}\dots x_2x_1)(x_{n-1}x_{n-2}\dots x_2x_1)$$

is a shortest directig word of $\mathbf{T}$. Since $|p| = \frac{n(n-1)}{2}$, then

$$d_{\mathcal{B}}(n) = \frac{n(n-1)}{2}.\blacksquare$$

In the proof of previous theorem, if $n \geq 3$ then the trap-directable automaton $\mathbf{T}$ is not aperiodic and not piecewise directable, that is, if $n \geq 3$ then $\mathcal{B}_n \nsubseteq \mathcal{AD}_n$ and $\mathcal{B}_n \nsubseteq \mathcal{PD}_n$.

Corollary 3 $d_{\mathcal{TD}}(n) = \frac{n(n-1)}{2}$.

Proof. By Theorem 2, it is obvious.

Example 4 *Let* $\mathbf{A} = (A, X, \delta)$ *be an automaton defined by*

$$A = \{1, 2, \dots, n\}, \quad X = \{x_1, x_2, \dots, x_{n-1}\}, \ (n \geq 3)$$

$$\delta(i, x_k) = k \quad (1 \leq i < k \leq n-1), \quad \delta(n-1, x_{n-1}) = n,$$

$$\delta(i, x_{i-1}) = i - 1 \quad (2 \leq i \leq n),$$

and moreover $\delta(i, x_s) = i$. *It is easy to see that* $\mathbf{A}$ *has a blocking state but* $\mathbf{A}$ *is not trap-directable.*

This means that, if $\mathcal{TD}_n$ $(n \geq 2)$ is the class of n-state trap-directable automata then $\mathcal{TD}_n \subset \mathcal{B}_n$. We note that $\mathcal{CD}_n \subset \mathcal{TD}_n$. (In the case $n = 2$, if $A = \{1, 2\}$, $X = \{x_1, x_2\}$, $\delta(i, x_1) = 1$ and $\delta(i, x_2) = 2$ $(i = 1, 2)$ then 1 and 2 are blocking states, thus the automaton $\mathbf{A} = (A, X, \delta)$ is not trap-directable.)

4 Subautomata and the Černý Conjecture

An equivalence relation ρ of the state set A of an automaton $\mathbf{A} = (A, X, \delta)$ is a *congruence* on $\mathbf{A}$ if

$$(a, b) \in \rho \Longrightarrow (ax, bx) \in \rho$$

for all $a, b \in A$ and $x \in X$. The ρ-class of $\mathbf{A}$ containing $a \in A$ is denoted by $\rho[a]$. The automaton $\mathbf{A}/\rho = (A/\rho, X, \delta_\rho)$ is called a *factor automaton* of $\mathbf{A}$ if

$$A/\rho = \{\rho[a];\ (a \in A)\} \quad \text{and} \quad \delta_\rho(\rho[a], x) = \rho[\delta(a, x)]\ (a \in A, x \in X).$$

If $\mathbf{A}' = (A', X, \delta')$ is a subautomaton of $\mathbf{A}$ then the congruence

$$\rho_{A'} = \{(a, b) \in A \times A;\ a = b \text{ or } a, b \in A'\}$$

is called the *Rees congruence of* $\mathbf{A}$ *induced by* $\mathbf{A}'$. The factor automaton $\mathbf{A}/\rho_{A'}$ is also denoted by $\mathbf{A}/\mathbf{A}'$ and it is called the *Rees factor automaton of* $\mathbf{A}$ *induced by* $\mathbf{A}'$. It is evident that $\mathbf{A}/\mathbf{A}'$ is a trapped automaton.

Denote $C(\mathbf{A})$ the set of all directed states of a directable automaton $\mathbf{A}$. If the automaton $\mathbf{A}' = (A', X, \delta')$ is a subautomaton of the automaton $\mathbf{A} = (A, X, \delta)$ then $C(\mathbf{A}) \subseteq A'$. If $C(\mathbf{A}) = A$ then we say that $\mathbf{A}$ is a *strongly directable automaton.*

We say that an automaton $\mathbf{A}$ is an *extension of an automaton* $\mathbf{A}'$ *by an automaton* $\mathbf{B}$ if $\mathbf{A}'$ is a subautomaton of $\mathbf{A}$ and $\mathbf{A}/\mathbf{A}'$ is isomorphic with $\mathbf{B}$. An automaton is directable if and only if it is an extension a strongly directable automaton by a trap-directable automaton ([6]).

Lemma 5 *If $\mathbf{A}' = (A', X, \delta')$ is a subautomaton of the directable automaton $\mathbf{A} = (A, X, \delta)$ such that $|A| = n$ and $|A'| = k$, then*

$$d(\mathbf{A}') \leq d(\mathbf{A}) \leq \frac{(n-k+1)(n-k)}{2} + d(\mathbf{A}').$$

Proof. If $\mathbf{A}' = (A', X, \delta')$ is a subautomaton of the directable automaton $\mathbf{A} = (A, X, \delta)$ then $\mathbf{A}'$ is also directable and $d(\mathbf{A}') \leq d(\mathbf{A})$. Furthermore $\mathbf{A}/\mathbf{A}'$ is a trap-directable automaton with the trap A'. If p and q are shortest directing words of $\mathbf{A}/\mathbf{A}'$ and $\mathbf{A}'$, respectively then $Ap \subseteq A'$ and so

$$|Apq| \leq |A'q| = 1.$$

This means that pq is a directing word of $\mathbf{A}$ and, by Corollary 3,

$$d(\mathbf{A}) \leq |pq| = |p| + |q| \leq \frac{(n-k+1)(n-k)}{2} + d(\mathbf{A}'). \blacksquare$$

Theorem 6 *If the Černý Conjecture holds for the class of all k-state directable automata, then it is holds for the class of all n-state $(n > k)$ directable automata containing at least one k-state subautomaton.*

Proof. Let $\mathbf{A} = (A, X, \delta)$ be an arbitrary n-state directable automaton and $\mathbf{A}' = (A', X, \delta')$ be an k-state subautomaton of $\mathbf{A}$ $(n > k)$. Assume that $d(\mathbf{A}') \leq (k-1)^2$. If $p \in X^*$ is a shortest directing word of $\mathbf{A}/\mathbf{A}'$ then by Corollary 3,

$$|p| \leq \frac{(n-k+1)(n-k)}{2}.$$

Let $q \in X^*$ be a shortest directing word of the subautomaton $\mathbf{A}'$. The word pq is a directing word of $\mathbf{A}$ and

$$|pq| \leq \frac{(n-k+1)(n-k)}{2} + (k-1)^2 \leq (n-1)^2.$$

From this it follows that, for every positive integer n, $d(n) \leq (n-1)^2$. ■

Lemma 7 *([8], [6])* $\mathbf{C}(\mathbf{A}) = (C(\mathbf{A}), X, \delta')$ *is the only strongly connected subautomaton of the directable automaton* $\mathbf{A} = (A, X, \delta)$.

In [17], I. C. Rystsov has proved the following theorem. We get this theorem from Lemma 7 and Theorem 6.

Corollary 8 *The Černý Conjecture holds for directable automata if and only if it holds for strongly directable automata.*

It is known that if $n \leq 4$ then the Černý Conjecture holds. In case $4 \leq n$ the best upper bound for the length of the shortest directing words is $\frac{n^3-n}{6} - 1$ ([15]). For some special classes of automata considerably better upper bounds are known.

Corollary 9 *The Černý Conjecture holds for the class of all n-state* $\mathbf{A} = (A, X, \delta)$ *directable automata for which* $|C(\mathbf{A})| < n$ *and*

$$n \leq 12 \quad or \quad 13 \leq n \quad and \quad |C(\mathbf{A})| \leq \frac{9 + \sqrt{12n + 49}}{2}.$$

Proof. By Lemma 7, $\mathbf{C}(\mathbf{A}) = (C(\mathbf{A}), X, \delta')$ is a subautomaton of the directable automaton $\mathbf{A} = (A, X, \delta)$. Let $|C(\mathbf{A})| = k < n$. If $k \leq 4$ then, by Theorem 6, the Černý Conjecture holds for $\mathbf{A}$.

Assume that $4 \leq k \leq n-1$. If $p \in X^*$ is a shortest directing word of $\mathbf{A}/\mathbf{C}(\mathbf{A})$ then, by Corollary 3,

$$|p| \leq \frac{(n-k+1)(n-k)}{2}.$$

If $q \in X^*$ is a shortest directing word of the subautomaton $\mathbf{C}(\mathbf{A})$ then

$$|q| < \frac{k^3 - k}{6}.$$

The word pq is a directing word of $\mathbf{A}$ and

$$|pq| < \frac{(n-k+1)(n-k)}{2} + \frac{k^3 - k}{6}.$$

But

$$\frac{(n-k+1)(n-k)}{2} + \frac{k^3 - k}{6} \leq (n-1)^2$$

if and only if

$$\frac{k(k-1)(k+1)}{6} \leq \frac{(n-1)(n+2k-4)}{2}.$$

It is easy to see that if

$$4 \leq k \leq \frac{9 + \sqrt{12n + 49}}{2}$$

then

$$\frac{k(k-1)(k+1)}{6} \leq \frac{(n-1)(n+2k-4)}{2}.$$

Furthermore if $n \leq 12$ then

$$n - 1 \leq \frac{9 + \sqrt{12n + 49}}{2}$$

and if $13 \leq n$ then

$$\frac{9 + \sqrt{12n + 49}}{2} \leq n - 1. \blacksquare$$

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