On the emergence of the Lorentzian metric structure of space-time in general relativity

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July 19, 2025

Abstract

In this short note we argue that, even if, as sometimes remarked, a Lorentzian manifold does not model correctly the structure of the spatio-temporal continuum as it is, yet a Lorentzian manifold should describe its macroscopic structure as we experience it.

More precisely, theoretically motivated by von Weizsäcker's chronological relative frequency interpretation of probability, and taking the Diaconis–Mosteller principle (also called the law of truly large numbers) as an empirical evidence in the macroscopic world, we argue that large collections of physical events appear in a composition of two fundamentally different patterns, termed as a progression and a sample here, making it unavoidable to use a Lorentzian-type metric on a manifold to describe matter-filled macroscopic regions of the spatio-temporal continuum.

PACS numbers: 01.55.+b; 04.20.Cv; 05.50.+q; 11.30.Cp

Keywords: *Lorentz metric; Diaconis–Mosteller principle; Probability*

1 Introduction

The *theory of general relativity* is an esteemed intellectual achievement within contemporary science, both æsthetically and scientifically. For this theory is not only beautiful thanks to its conceptional simplicity combined with inspiring predictive power and mathematical elegance, it is also in a century-old perfect accordance with independent physical experiences, extending from accurate terrestrial laboratory and satellite experiments toward sophisticated deep space astronomical observations.

Apparently Einstein's theory correctly captures our impressions about time and space at meso- and macroscopic scales, including their conceptional similarity composed with their experiential difference. This co-existing similarity-difference of time and space enters the general relativity formalism as a fragile and tensionful balance between structures. It is declared that time and space exist in general relativity and they together comprise an $1+3=4$ dimensional physical continuum: the *space-time*, mathematically modeled by a pair (M,g) consisting of a 4 dimensional *differentiable manifold* M, which grasps the similarity by its homogeneity, and an $(1,3)$-type or *Lorentzian metric* g on it, which

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certainly grasps the difference by its unusual signature. However (M,g) is subject to Einstein's field equations which disturbes this simple division, for general relativity in this way becomes an inherently timeless theory, known as the "problem of time" (cf. [24, Appendix E] for a technical exposition, and e.g. [2, Part 3], [5, Chapter 2] for broader physical and philosophical surveys). As a result, at a deeper layer of the theory the balance is lost rendering the role played by the Lorentzian signature of the metric less clear.

Apart from this, there has been a long and diverse debate in the scientific community concerning the nature and relevance of the co-existing similarity-difference of time and space. In the standard physics literature the emphasis is mainly put on similarity (except perhaps in classical thermodynamics and quantum measurement theory) in a well-known systematic way; while the much more complicated problem of difference has been approached from various directions (cf. e.g. [2, 4, 5, 6, 12, 16, 20, 25]) with yet only partial success. Nevertheless these efforts tend to reach at least one common insight, namely that difference between time and space gets more-and-more significant as the number of effectively available degrees of freedom in a physical system increases. This not necessarily means being macroscopic, as for instance the quite different roles played by time in large thermodynamical systems and in celestial mechanics shows.

Returning to the mathematical formalization of general relativity, let us make a (certainly incomplete) list of problems, comments, questions, either physical or mathematical in their characters, triggered by the basic assumption of general relativity, that the physical spatio-temporal continuum is modeled mathematically by a Lorentzian 4-manifold (M,g). Modeling the physical continuum itself with the mathematical one M has already its own serious difficulties [1, 26], however let us rather focus here on puzzles stemming from the utilization of a Lorentzian-type metric g. In what follows we list items extending from physical toward mathematical issues, however doing so we do not mean an ordering by relevance. In our opinion, when treating difficulties in a physical theory, both conceptional-physical and technical-mathematical problems have an equal importance.

1. *Physical interpretation of Lorentzian geometric concepts in vacuum.* Beyond the trivial vacuum solution which is the flat Minkowskian space-time, the vacuum Einstein equation with or without cosmological constant, admits non-trivial i.e. curved solutions too. An example is the Schwarzschild geometry describing a static and spherically symmetric configuration of pure gravity i.e., not shaped by any form of gravitating matter, rather forced alone by the unavoidable interaction of the gravitational field with itself. (Mathematically the self-interaction of gravity is reflected in the inherent non-linearity of the Einstein equation.) However even in this temporally static and spatially empty situation for example, the space-time is furnished with additional structures according to the *Lorentzian paradigm*. The most important extra structure is the collection of light cones (closely related to the causal structure). Geometrically, the two light cones having a common vertex at a space-time point emerge by tracing all null curves in space-time arriving at or departing from that space-time point. But what is the physical description of these curves? The most basic physical assumption of general relativity is that space-time is a union of its points and these represent "elementary physical events", and accordingly the corresponding light cones are physically realized by "light beams" absorbed or emitted at these elementary physical events [8]. But thinking physically, what kind physical happening can occur in a space-time point in the completely static vacuous physical situation modeled by the Schwarzschild geometry? In particular, how light beams can be absorbed and emitted at these points or exist here at all i.e., in an overall physical situation described by the static-empty Schwarzschild geometry? Operationally it seems light cones should appear only in those space-times which contain electromagnetic radiation, like e.g. the Kerr–Newman space-time, or at least some other propagating massless field. (Perhaps they can be introduced in space-times describing pure gravitational radiation too, but in this case some care is needed because the physical characterization of light cones seems to be self-referential.)

2. *Problems with the zero distance between distinct physical events.* Thinking in plain geometric terms the two light cones having a common vertex at a space-time point consist of those *distinct* space-time points whose Lorentzian distances from the given point are precisely zero. However thinking physically again, this is a strongly counter-intuitive feature of the *Lorentzian paradigm*. As Cornelius Lanczos puts it in his 1974 book on Einstein [14, p. 32]:

> [...] Accepting the paradoxical situation concerning the 'zero distance', we must be consistent and accept the leading principle of the Gauss–Riemann type of geometry, called 'metrical geometry', that the distance is *the* quantity which decides all other geometrical properties. Now let us consider the following physical situation. It seems reasonable to assume that 'zero distance' means strong physical interaction. We now consider an atom emitting light in the Andromeda Nebula. The light reaches Earth after three million years and interacts with the observing eye at the end of the telescope. However, the *four dimensional distance between the atom and the eye was zero throughout the three million years of travel*. Why then is the interaction restricted to the end of the journey, when the decisive geometrical quantity remains all the time zero, without any change?
>
> I had a chance to discuss this question with Einstein who admitted its seriousness and felt very uncomfortable with an indefinite line element. Yes, he said, the indefinite metric offers a great puzzle which must arise from some deep seated source. But for the time being he did not see a solution and was willing to take the difficulty temporarily in his stride.

Working with general relativity nowadays, we already live in comfort with this fact and consider it as a subtlety or oddity of the widely accepted mathematical formulation of general relativity. But Lanczos continues with a warning [14, pp. 32-33]:

> It seems to me that if we accept an apparent irrationality on empirical grounds, we will have to pay penalty sooner or later in heavy currency, as we had to pay the price for Newton's 'absolute space' and 'absolute time', and the equivalence of heavy and inertial mass. If we encounter an apparent irrationality at the very beginning of our speculations, we must feel permanently on dangerous ground.

As a result of this critical attitude Lanczos released his own approach to the problem of indefiniteness, using classical field theoretic arguments [13, 15]: he supposes that the so-called classical vacuum is rather a composition of highly excited states and the originally Riemannian metric of the space-time continuum switches to a Lorentzian one only by averaging over these excited states, and the Lorentzian signature perhaps appears at macroscopic levels only.

3. *Ellipticity lost.* Partial differential equations (PDE's) are classified broadly as elliptic, parabolic and hyperbolic, exhibiting completely different behaviours and accordingly, requiring very different mathematical tools to handle them. For instance elliptic problems possess a very pleasant property called *elliptic regularity* leading, at least in the linear case, to a satisfactory theory regarding existence, uniqueness and differentiability of their solutions. Another extraordinary important tool in dealing with linear elliptic problems is the *Atiyah–Singer index theorem* connecting analysis and topology at a very deep mathematical level. The most relevant fundamental differential equations of physics (like the Dirac, the wave, the gauge-fixed Yang–Mills, the linearized Einstein equations, etc.) are elliptic in Riemannian signature however are *not* in Lorentzian one. Consequently, solving PDE problems in Lorentzian signature is often non-tracktable simply by lacking poweful mathematical techniques, which is in sharp contrast to the Riemannian case. Perhaps this is the right place to mention the various other mathematical difficulties arising from Lorentzian signature, but not necessarily related with ellipticity (like the non-compactness of the orthogonal group, extra difficulties in path integration, etc., etc.), too.

4. *Bad category.* It is attributed to the French mathematician Alexander Grothendieck the saying that "It is better to have a good category with bad objects than a bad category with good objects." Whatever it precisely means, this is a summary of mathematicians' old experience that only those mathematical things are comfortable to work with, of whose category can be completed with respect to category theoretic (co)limits, by adding only "reasonably singular" limiting i.e., ideal objects to the original ones. Most of compact manifolds cannot carry Lorentzian metrics (the obstruction is the manifold's Euler characteristic [22, Theorem 40.13] which is not trivial in even dimensions) and this fact, together with the failure of the *Hopf–Rinow theorem* here (cf. [3, p. 4] makes the theory of *global Lorentzian geometry* quite vulnerable against counterexamples. Indeed, since most of Lorentzian manifolds are by default not complete, all kind of counterexamples can easily be constructed by artificially cutting out subsets (for an excellent summary see [3]). However one has the permanent feeling that most of these counterexamples are somehow irrelevant from a "natural" or "physical" point of view. This "openness" toward counterexamples indicates that the category of Lorentzian manifolds cannot be completed nicely in the above sense. In sharp contrast to this situation, the category of compact Riemannian manifolds is a good category because compactness is not an obstacle at all concerning Riemannian metrics hence the Hopf–Rinow theorem guarantees completeness of individual examples; consequently counterexamples are not so easy to find. Moreover, there is a satisfactory general limit construction in this category too, namely the *Gromov–Hausdorff convergence* of compact Riemannian manifolds.

5. *The Witt algebra.* From the abstract perspective of the general mathematical theory of finite dimensional inner product spaces over a fixed ring [17], definite inner product spaces are distinguished in the following sense. Considering the relevant case of $\mathbb{R}$ here only, one can introduce certain equivalence classes of real vector spaces V carrying scalar products i.e. non-degenerate symmetric $\mathbb{R}$-bilinear forms $g : V \times V \to \mathbb{R}$. The tensor-product-over-$\mathbb{R}$ and the orthogonal-sum operations descend to these so-called Witt classes yielding a very important commutative ring with unit, the *Witt ring* $W(\mathbb{R})$ of the reals (for a precise definition cf. [17, Definition I.7.1]). The signature of a scalar product is well-defined in every Witt class and in fact (see [17, Corollary III.2.7]) it gives rise to an isomorphism $W(\mathbb{R}) \cong \mathbb{Z}$. However one can prove (see [17, Theorem III.1.7]) that every element of the Witt ring over $\mathbb{R}$ i.e., every Witt class of real scalar product spaces, contains a *canonical* representative, namely the unique space in this class which carries a *definite* scalar product. Consequently the aformentioned isomorphism arises simply by computing the signature of the unique definite scalar product in each Witt class whose absolute value is therefore equal to the dimension of the underlying real vector space. That is to say, up to a natural equivalence in the theory inner product spaces, in a given dimension there exist only two scalar product spaces over the reals, namely the ones which carry fully positive or negative definite scalar products.

By these (and probably furthers, to be added by the Reader) heterogeneous remarks one might feel uncomfortable with Lorentzian geometry and ask whether or not a Lorentzian manifold only a pseudo-structure in physics is. That is, perhaps it does not model any fundamental feature of the pure spatio-temporal continuum shaped by gravitational force alone. However, even if this is the case, Lorentzian structure can still be relevant as an emergent macroscopic phenomenon, caused by the "contamination" of the pure spatio-temporal continuum with macroscopic matter, which actually the real macroscopic physical situation is. In the remaining part of this note we make an attempt to bring this idea to a more solid basis, by evoking an interesting principle of applied statistics, and use it as an empirical argument to support the utilization of Lorentzian geometry at least at macroscopic levels.

2 A conceptional analysis of the Diaconis–Mosteller principle

The world surrounding us, apparently, perhaps as a manifestation of the *ergodic hypothesis*, has the tendency to realize the even most incredible but physically allowed situations. Studying the exciting and diverse problem of occurences of surprising events, extraordinary coincidences in daily life, Persi Diaconis and Frederick Mosteller introduced a remarkable principle in 1989, termed as the *law of truly large numbers*, which in its full generality can be formulated as follows [7, p. 859]:

With a large enough sample, any outrageous thing is likely to happen.

This principle offers a simple rational (but perhaps emotionally not satisfying) explanation how unlikely things can come to existence in the macroscopic world. Our goal is to draw some physical conclusions from this principle; however before that we have to clarify some terms in its formulation.

The first of these terms is an *outrageous thing* what we immediately translate to the more neutral language of science as a *low probability physical event*, and then ask ourselves what *probability* is. Instead of diving into this bottomless topic [11] here, we departure from von Weizsäcker's chronological approach [25] that probability is a mathematical way to model an obvious aspect of *time*, namely that future consists of *possibilities* hence is objectively not known in the present (on the contrary, the past consists of *facts* hence is known). Consequently in general one can make only *predictions* on *forthcoming* physical events. We accept therefore that probability is the *predefined relative frequency* and in this way probability is an objective aspect of reality. Note the subtlety that this chronological approach is free of the usual *circulus vitiosus* in the plain frequentist definition, for it supposes that probability can be (somehow) determined, in principle unambigously, in advance, and also supposes that any particular empirical sequence of relative frequencies in the corresponding truncated series of already-happened-physical-events, will converge to this known abstract real number; however, as a price, the exact convergence never can be confirmed in any finite experiment. We also accept that the mathematical formalization of classical probability is Kolmogorov's theory. Now we can make our

Assumption 1. (Existence and uniqueness of probability) *Probability, as the limit of a sequence of relative frequencies in a progression of independent physical copies of a model event, is a well-defined scalar taking values in* $[0,1] \subset \mathbb{R}$*. That is, this real number exists and is independent of the particular observer who records the events, makes unbiased judgements about their favourability, and then computes relative frequencies and their limit in the progression.*

Here by a *progression* we mean an at most countable set E of physical events, which for an observer O appear in an ordering $<_O$, might be called as "strictly later than", according to the observer's proper time (i.e., simultaneous detection of events is not permitted). Then, as usual, the *relative frequency* of the detected and accordingly ordered truncated set of events $e_1 <_O e_2 <_O \cdots <_O e_n$ taken from E is the ratio $\frac{|\text{favourable cases}|}{n} \in [0,1]$. Although for an objectively given E the ordering of events, the judgements on their favourability, the values of the corresponding relative frequencies, etc., etc., can depend on the particular observer at finite n's, it is assumed that whenever $n \to +\infty$ the limit exists unambigously and independently of any particular *ordinary* observer, giving rise to the probability $p \in [0,1]$ which is therefore unambigously assigned to E.

Next we turn to the *likely to happen* term in the above daily formulation of the Diaconis–Mosteller principle. The precise formulation of this idea in terms of probability leads to our

Assumption 2. (Diaconis–Mosteller principle) *The probability that an event occurs within a sample of independent physical copies of a model event having positive probability, gets arbitrarily close to* $1 \in [0,1]$ *as the finite cardinality of the sample increases.*

Here *a fortiori* by a *sample* we mean that sort of collection of physical events for which the formulated probabilistic behaviour, as an empirical evidence for an *ordinary* observer O, holds. However note that in this way two similar expressions come into use: the “progression” and the “sample” of physical events. Both of them certainly label specific collections of physical happenings in **Assumption 1** and **Assumption 2** respectively, however their distinction is not clear. The following simple but key technical lemma demonstrates that they are in fact strictly different, hence validating the terminology:

Lemma 2.1. *Suppose that* **Assumption 1** *and* **Assumption 2** *are valid. Take a collection* $\{e_1,\dots,e_k\}$ *consisting of* $0<k<+\infty$ *independent physical events with uniform probability* $0<p<1$. *Provided the cardinality k is large enough,* $\{e_1,\dots,e_k\}$ *is strictly either a progression or a sample, if any.*

Proof. Since the complementers of independent events are also independent, the probability that an event with probability $0\leqq p\leqq 1$ does *not* happen in k independent trials is $(1-p)^k$ hence the probability that it *does* happen in the collection $\{e_1,\dots,e_k\}$ is $q=1-(1-p)^k$. Suppose **Assumption 2** holds and $\{e_1,\dots,e_k\}$ is a sample; then $q\to 1$ as $k\to+\infty$. Suppose that **Assumption 1** also holds and $\{e_1,\dots,e_k\}$ is also a progression and $0<p<1$ is the relative frequency of favourable events; then $q\approx 1$ for large k implies that in this case in fact $p\approx 1/k$ that is, there exists a constant $0<c<+\infty$ independent of k satisfying $0<p\leqq c/k<1$ for sufficiently large k's. However in this case

$$q=1-(1-p)^k\leqq 1-\left(1-\frac{c}{k}\right)^k\leqq 1-\frac{\mathrm{e}^{-c}}{2}<1$$

for large k's contradicting that $q\to 1$ as $k\to+\infty$. Thus a sufficiently large collection with uniform probability $p\neq 0,1$ cannot be both a progression and a sample. □

Consequently Lemma 2.1 forces us to distinguish progressions and samples from each other. One can argue that, as an aspect of macroscopic reality, large ensembles of physical events arrange themselves phenomenologically for an observer in two fundamentally different manners. One is characterized by **Assumption 1**, in which an ensemble exhibits a more-and-more sharp but otherwise arbitrary probability $p\in[0,1]$ as its cardinality increases, hence the ensemble approaches a purely probabilistic description; this is what we call a *progression* (but also might be termed as a *timelike arrangement*). The other is characterized by **Assumption 2**, in which all probabilities behave like $q\to 1$ as cardinality increases, hence the ensemble approaches a purely deterministic description, this is what we call a *sample* (but also might be termed as a *spacelike arrangement*). Moreover, **Assumption 1** also guarantees that this distinction is independent of the particular *ordinary* observer. A general arrangement of a large ensemble is then a mixture of the two extreme cases i.e., is a mixture of a progression and a sample.

Before proceeding further we would like to clarify that in our understanding this observation offers an argument on an empirical (i.e. not metaphysical, or mathematical, or theoretical physical, psychological, etc.) ground that in the continuum of physical events there exist “mainly probabilistic” direction(s) as well as “mainly deterministic” direction(s) at least macroscopically. From this angle our approach is similar to that of Callender who argues for the existence of a distinguished, “timelike” direction in space-time using rather mathematical, especially algorithmical compressibility arguments, see [5, Chapters 6-8]. However rearranging the building blocks of this argumentation we can also say: the empirical evidence summarized in the Diaconis–Mosteller principle (**Assumption 2** here) and the mathematical fact expressed in Lemma 2.1 imply that probability and cardinality are *independent* data of a large ensemble of physical events. Probability characterizes the functional or temporal distribution (in accord with von Weizsäcker's chronological relative frequency interpretation [25, Teil II.2] and summarized in **Assumption 1** here), while cardinality characterizes the structural or spatial distribution of a large ensemble of physical events.

3 Recovering the Lorentzian structure

As a next and plain technical step, we work out a mathematical model for the continuum of physical events which turns out to be nothing else than a Lorentzian manifold. This can be considered as an analogue of the space-time reconstruction carried out e.g. in [8] however goes along conceptionally quite different lines.

First consider a discrete picture. Let $\{e_1,\dots,e_m\}$ be a sample as in **Assumption 2** consisting of m independent physical events of equal *a priori* probability $0<p<1$. The probability q that the event occurs in the sample thus satisfies $q\to 1$ as $m\to+\infty$. For every $e_i\in\{e_1,\dots,e_m\}$ in this sample put $e_{i,1}:=e_i$ and take the corresponding progression $\{e_{i,1},e_{i,2},\dots,e_{i,n}\}$ for this event as in **Assumption 1** whose relative frequency therefore converges to $p=p(e_{i,1})$ whenever $n\to+\infty$. Lemma 2.1 makes sure that $\{e_{1,1},\dots,e_{m,1}\}\neq\{e_{i,1},\dots,e_{i,n}\}$ for every $i=1,\dots,m$ i.e., the sample differs from all progressions of its elements as well as by the independency of events it is obvious that $\{e_{i_1,1},\dots,e_{i_1,n}\}\cap\{e_{i_2,1},\dots,e_{i_2,n}\}=\emptyset$ for every $1\leqq i_1<i_2\leqq m$ i.e., the progressions are disjoint. In this way we roughly obtain an $m\times n$ grid $E_{m,n}:=\{e_{i,j}\}_{\substack{i=1,\dots,m\\ j=1,\dots,n}}$ of events.

Now we would like to pass to the continuum limit, more precisely carry out this construction in a continuum of physical events too. Consider a finite dimensional connected compact oriented Riemannian manifold (M,g). Evoking here our previous constructions in [9, Section 4] we recall that (M,g), conventionally considered as a rigid geometric structure exhibiting geodesics, curvature, etc., only, in fact canonically gives rise to a Kolmogorov probability measure space $\left(M,\mathscr{A}_g,p_g\right)$, too. Indeed, the orientation of M together with the Riemannian metric g induces a measure μ_g on M; then $\mathscr{A}_g$ is defined to be the σ-algebra consisting of all μ_g-measurable subsets of M, including M; then writing $V_g:=\int_M\mu_g$ satisfying $0<V_g<+\infty$ for the volume of (M,g) and putting $p_g:=\frac{1}{V_g}\mu_g$ we obtain a probability measure given by $p_g(A)=\frac{1}{V_g}\int_A\mu_g$ for $A\in\mathscr{A}_g$. It is remarkable that this probability space up to diffeomorphisms is independent of the particular metric g used to construct it; for it follows from a theorem of Moser [18] that if (M,h) is another Riemannian space yielding $(M,\mathscr{A}_h,p_h)$ then there exists a fixed orientation-preserving diffeomorphism $f:M\to M$ such that if $A\in\mathscr{A}_g$ then $f(A)\in\mathscr{A}_h$ and $p_g(A)=p_h(f(A))$. Proceeding further, note that if $B\in\mathscr{A}_g$ then $q_g(B)=\frac{1}{V_g}\int_B\mu_g\to 1$ whenever $\int_B\mu_g\to V_g$, resembling the probability-cardinality relationship in **Assumption 2**. Given an *ordinary* observer O, we can therefore think of the manifold M as a non-countable analogue of a large sample having regularized infinite cardinality equal to its volume, and think of the geometrically-given probability p_g as for some $A\in\mathscr{A}_g$ the number $0\leqq p_g(A)\leqq 1$ is the probability that an elementary physical event appears in this prescribed measurable region i.e., if $e\in M$ then $e\in A$ (cf. [9, **Assumption-physical form**]). Hence we can in effect identify $e\in M$ with its appropriately small neighbourhood $e\in U\subseteqq M$ and write $p(e):=p_g(U)$ for the corresponding probability in accord with the notation above. By **Assumption 1** we have to assign a progression of events to every $e\in M$ such that the relative frequencies in this progression converge to the *a priori* probabilities $0\leqq p(e)\leqq 1$ just constructed. Now suppose that strictly $0<p(e)<1$. In this case Lemma 2.1 forces that the sought progression is not a subset of M. Therefore take another differentiable manifold N and a map $\pi:N\to M$ such that $\pi^{-1}(e)\subset N$ is the progression for $e\in M$ as detected by the *ordinary* observer O at e. Since M locally looks like $\mathbb{R}^k$ we also suppose roughly that $\pi^{-1}(e)\cong\mathbb{R}^\ell$. However the demand that $\pi^{-1}(e)$ admits an ordering $<_O$ forces to take $\ell=1$ and globally we have to suppose that N is oriented and $\pi:N\to M$ is orientation-preserving; moreover $0<p(e)<1$ implies for its neighbourhood that $e\subsetneqq U\subsetneqq M$ and we accepted that $e\in M$ has been effectively identified with U. Consequently we more precisely suppose that the local triviality property of vector bundles holds i.e., $\pi^{-1}(U)\cong U\times\mathbb{R}$ such that $\pi^{-1}(x)\cong\mathbb{R}$ is a linear isomorphism for every $x\in U$. The uniqueness part of **Assumption 1** i.e., the assumption that probability is independent of a particular *ordinary* observer who records the corresponding relative frequencies, finally makes sure that the resulting structure of N is independent of the particular observer O who was used to construct it. In this way we can quite naturally conclude that N has the structure of an oriented real line bundle that is, an oriented rank-1 real vector bundle over a compact manifold M. From now on we embed M into N by the zero section as usual.

The discrete and the continuum pictures are unified by taking an inclusion $E_{m,n} \subset N$. We suppose that it is sample-preserving i.e., $E_{m,n} \cap M = \{e_{1,1}, \dots, e_{m,1}\}$ is the sample part of $E_{m,n}$; however we do not demand this embedding to be progression-preserving as well i.e., for an event $e_{i,1} \in E_{m,n} \cap M$ its progression $\{e_{i,1}, \dots, e_{i,n}\} \subset E_{m,n}$ is not necessarily included in $\pi^{-1}(e_{i,1}) \subset N$; this captures the possibility that the progression is distant to an observer who records it. Nevertheless we can think of the discrete subset $E_{m,n}$ as a collection of physical events which are distinguished by observation within an ambient continuum N of physical events. Finally we make $E_{m,n}$ macroscopic by sending $m \to +\infty$ and $n \to +\infty$ (hence $mn \to +\infty$) such that $E_{m,n} \subset N$ converges to an everywhere dense countable subset $E \subset N$.

Pick an event $e \in E \cap M \subset N$ and consider the corresponding tangent space T_eN. The fibre bundle structure $\pi : N \to M$ satisfying $\pi^{-1}(e) \cong \mathbb{R}$ induces $T_eN = T_eM \oplus T_e\pi^{-1}(e) \cong T_eM \oplus \mathbb{R}$ and recall that there already exists a partially defined positive definite scalar product $g_e : T_eM \times T_eM \to \mathbb{R}$. It is therefore natural to extend g_e to an entire symmetric non-degenerate bilinear form $h_e : T_eN \times T_eN \to \mathbb{R}$. In light of the Jacobi–Sylvester theorem [17, Theorem III.2.5] this extension is uniquely characterized by the signature of h_e. To find the signature, consider two smooth curves $\gamma, \delta : \mathbb{R} \to N$ satisfying $\gamma(0) = \delta(0) = e \in E \subset N$ such that, according to an *ordinary* observer O, the curve γ connects events in $E \subset N$ which belong to a sample while δ connects events in $E \subset N$ which belong to a progression of e (see the left side of Fig. 1). Referring to Lemma 2.1 which says that samples and progressions are necessarily different, we can suppose that $\dot{\gamma}(0)$ and $\dot{\delta}(0)$ are linearly independent within T_eN (see the right side of Fig. 1).

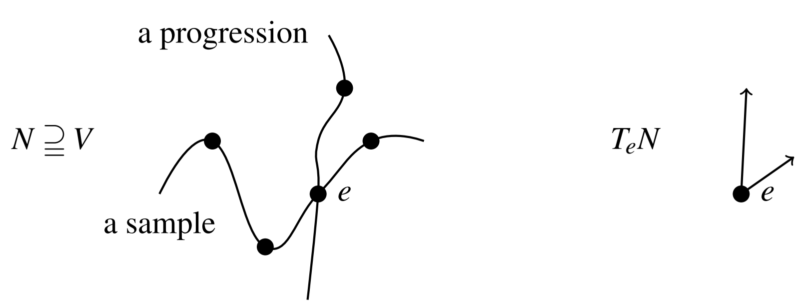

Figure 1. The local (left) and the infinitesimal (right) shape of a large progression and a large sample through an event e in the ambient continuum N of physical events.

Take an orientation-preserving diffeomorphism $\Phi : N \to N$ such that $\Phi(e) = e$ and the corresponding derivative $\Phi_*(e) : T_eN \to T_eN$ is an h_e-orthogonal transformation. Recalling the usual distinction between gauge and symmetry transformations[1] we can interpret Φ as a *symmetry transformation at* $e \in N$. The uniqueness part of **Assumption 1** i.e., that probability is an invariant scalar, implies that

[1]Given a space-time (N,h) a general element $\Phi \in \mathrm{Diff}^+(N)$ is a *gauge* transformation i.e., (N,h) and $(\Phi(N), \Phi^*h)$ are considered physically the same space-times; however if in addition $\Phi^*h = h$ holds i.e., $\Phi \in \mathrm{Iso}^+(N,h) \subsetneqq \mathrm{Diff}^+(N)$ then Φ is called a *symmetry* transformation i.e., (N,h) and $(\Phi(N),h)$ are considered physically different, cf. e.g. [24, p. 438].

samples and progressions cannot be transformed into each other by Φ; consequently $\Phi_*(e)\dot{\gamma}(0) \neq \dot{\delta}(0)$ and likewise $\Phi_*(e)\dot{\delta}(0) \neq \dot{\gamma}(0)$. Thus the orbit of $\dot{\gamma}(0)$ under the group of all h_e-orthogonal transformations cannot contain $\dot{\delta}(0)$ and *vice versa*. However this readily implies that this group does not act transitively on rays of tangent vectors hence the non-degenerate symmetric bilinear form h_e on T_eN is necessarily *indefinite*. We know that $h_e|_{T_eM} = g_e$ is positive definite and $T_eM \subset T_eN$ is a subspace having codimension 1 thus we conclude that h_e has a Lorentzian signature. Repeating these considerations over all points of the everywhere dense subset $E \subset N$ and assuming smoothness we extend h_e over N and come up with a Lorentzian manifold (N,h) as claimed.

Using our two plausible probability assumptions of Section 2 we could equip the spatio-temporal continuum fullfilled with macroscopic matter with the expected structure of a Lorentzian manifold, called a space-time in general relativity in the broad sense. More precisely we could reproduce the signature of the metric i.e., its infinitesimal structure only. The next question is whether or not can one go further and say something on the local and then the global structure of h that is, reproduce the Einstein equation in this framework over N too and then obtain a space-time in the strict sense i.e., a Lorentzian manifold satisfying a specific Einstein equation. (Note that the treatment of the Einstein equation in the substantiation of general relativity in [8] is missing.) Concerning this important point, observe that our constructions so far suffer from an ambiguity. Given an event $e \in M \subset N$, on the one hand we have constructed its probability from the spatial metric g along its sample M as $p_g(U)$ where $e \in U \subset M$ is an open subset; on the other hand **Assumption 1** says that the probability $p(e)$ of this event is related with its progression in N. We also know via Lemma 2.1 that the progression of e cannot be a subset of $M \subset N$ hence the asserted equality $p_g(U) = p(e)$ assumes some relationship between the 1 codimensional Riemannian submanifold (M,g) and its ambient space (N,h). One expects that this relationship is the Einstein equation. Although we have not been able (yet) to derive precisely the Einstein equation itself in this way, we can at least collect some promising evidences in this direction. Introducing the extrinsic curvature k of $(M,g) \subset (N,h)$ we can write the Einstein equation on (N,h) with matter T, satisfying the dominant energy condition, in the form of constraint equations along $(M,g) \subset (N,h)$ as usual, cf. [24, Equations 10.2.41 and 10.2.42]:

$$\begin{cases} s_g - |k|_g^2 + \mathrm{tr}^2 k = 16\pi\rho, \\ \\ \mathrm{div}(k - (\mathrm{tr}\, k)g) = 8\pi J, \\ \\ \rho \geq |J|_g \geq 0. \end{cases}$$

In the Riemannian m-manifold (M,g) the small-radius-expansion of the volume of a ball of radius r about a point $x \in M$ exists and looks like $\mathrm{Vol}_g B(x,r) = \frac{\pi^{\frac{m}{2}}}{\Gamma(\frac{m}{2}+1)} r^m \left(1 - \frac{s_g(x)}{6(m+2)} r^2 + \dots\right)$. Inserting the first of the Einstein constraint equations only (i.e., the full Einstein equation as well as the dominant energy condition is not used), the local probability can therefore be written as

$$p(e) = p_g(B(e,r)) = \frac{1}{V_g} \frac{\pi^{\frac{m}{2}}}{\Gamma(\frac{m}{2}+1)} r^m \left(1 - \frac{16\pi\rho(e) - \mathrm{tr}^2 k(e) + |k(e)|_g^2}{6(m+2)} r^2 + \dots\right)$$

indicating that $p(e)$ is *not* determined fully by the pointwise matter energy density $\rho(e)$ with $e \in M$ alone, but depends on the local matter distribution about $e \in N$ in the temporal direction too through the extrinsic curvature tensor k and the small radius $0 < r < \varepsilon$. This picture is at least consistent with the chronological interpretation of probability in **Assumption 1**.

4 Conclusion and outlook: dicing around a black hole

In this paper we have exhibited an argument to support the utilization of Lorentzian metrics in general relativity, despite the known physical and mathematical problems related with the inconvenient signature of these type of metrics. However to close we would like to call attention to a strong theoretical limitation of our approach related with the unambigouos extendibility of the concept of probability to space-times possessing complicated causal structures [21].

Throughout the text and especially around the formulation of **Assumption 1** and **Assumption 2** we have carefully emphasized that an *ordinary* observer's experiences are used to substantiate the whole set-up here. However accepting that general relativity describes not only our mild terrestrial experiences but works correctly in the dynamical and strong range of gravity too (which is supported by the recent direct gravitational wave and black hole observations) the important question arises how to interpret probability in these more general situations. Recall that in our framework probability is the predefined relative frequency of a sequence of events (called progression here); this abstract real number then can be verified by an observer with certain accuracy (but never exactly) by detecting these events and counting the favourable cases. We have assumed that probability in this way is a well-defined scalar. Acknowledging this observer-dependence our worry can be put as follows: *in what extent is probability a relativistically invariant scalar quantity?* Before targeting this problem we note that there are other approaches to probability which are apparently more safe because do not refer to any observer. Consider for instance the so-called *classical interpretation* by Laplace via symmetry considerations [11]: the probability of getting the number 6 in one toss with an unloaded dice is $\frac{1}{6}$ precisely because this dice is a *perfect cube* having 6 equal sides, etc. This elegant forecasting of probability is then indeed verified (with finite accuracy) in a sequence of events. However note that in this empirical verification process we have assumed by convention that the dice and the gamer more-or-less stay together in space what we call an *ordinary* situation. Instead of this imagine for example that the dice is on the board of a distant spaceship moving, with respect to the gamer, with a velocity comparable with the speed of light. Apparently the shape of the dice is distorted by Lorentz contraction hence the gamer concludes that the probability is not equal to $\frac{1}{6}$; meanwhile receiving a message with the result of a long sequence of trials, recorded by the ship's crew, will support $\frac{1}{6}$. Thus it seems the classical and the relative frequency interpretations of probability diverge already in a simple special relativistic situation. This conclusion is however wrong: in our context very surprising computations [19, 23] demonstrate that (as long as a single eye is used and the image occupies a small solid angle) the effect of Lorentz contraction on the visual appearance of a solid body is not distortion but rather a perfect rotation! Hence quite interestingly spatial symmetry is preserved at least in the realm of special relativity. However imagining more wild games (dicing around a neutron star, or in the vicinity of a rotating-oscillating black hole whose space-time geometry is even not known yet) one would not expect a similar special relativistic "conspiration" which could save the coherence of various probability concepts.

Thus returning to **Assumption 1** we ask then again in what extent probability an observer-independent scalar quantity is. Imagine the situation inspired by [10]: consider two observers, one is orbiting on a stable equatorial orbit around a massive slowly rotating Kerr black hole while the other is a gamer holding an unloaded dice and departing into the black hole without hitting its central singularity (hence in principle can survive the trip). According to his own clock, he will cross the (outer) event horizon in finite time and after this event he starts tossing the dice. Meanwhile the gamer will likely conclude that the relative frequency approaches the usual $\frac{1}{6}$ probability, the observer outside the black hole and watching the game surely cannot reach a definitive answer ever. Hence this bit awkward example indicates that even **Assumption 1** alone might break down in truely generic relativistic situations.

Declarations and statements. No any form of AI was used to write this article. All not-referenced results in this work are fully the author's own contribution. There are no conflict of interest to declare that are relevant to the content of this article. The work meets all ethical standards applicable here. No funds, grants, or other financial supports were received. Data sharing is not applicable to this article as no datasets were generated or analysed during the underlying study.

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