Representations of an operator algebra and the geometry of four-manifolds

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Abstract

In this paper, using connected closed oriented smooth 4-manifolds, some representations of the hyperfinite II_1-type factor von Neumann algebra are constructed. The Murray–von Neumann coupling constant of these representations gives rise to a new real-valued smooth 4-manifold invariant whose very first properties are investigated here. These will enable us to relate the possible invalidity of the 4 dimensional smooth Poincaré conjecture to the possible existence of a certain sequence of smooth 4-manifolds having finite fundamental groups.

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1 Introduction and summary

Compared with the others, the *hyperfinite* II_1 *factor von Neumann algebra* has a proliferation of representations: the moduli space of its non-equivalent representations is isomorphic to $\mathbb{R}_+$ (and accordingly, the II_1-type is the only one among factors whose K_0 group is non-trivial, namely isomorphic to $\mathbb{R}$). The aim of this paper, as a substantial extension of our previous efforts in [7, 8] in this direction, is to reconsider the following common sense argument: despite the existence of many non-trivial inequivalent representations of the hyperfinite II_1 factor, from a mathematical viewpoint only one of them appears as "reasonable", namely its *standard representation*. This is because one can argue that this representation is the basic and most important one, for other representations, if cyclic, arise from the standard one via the *Gelfand–Naimark–Segal construction* but using quite irrelevant states on the algebra, and all the rest are direct sums of these; or one can also argue mathematically that all representations can be obtained uniformly by simply amplifying the standard representation. However, recalling that in physics in general and in relativistic quantum field theories in particular, the existence of inequivalent representations of one and the same algebra of observables bears a deep meaning (*e.g.* in connection with phase transitions of the corresponding physical system, cf. [4, 11]), one can also say that the

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aforementioned general mathematical descriptions of representations of the II_1 hyperfinite factor, in fact their plain reduction to the unique standard one, might overlook some essential features.

Approaching the question this way, one can then ask again whether or not the other representations of the hyperfinite II_1 factor bear some relevance too. Our first result towards constructing *contentful* new representations connects the general theory of modules over the hyperfinite II_1 factor with four dimensional differential geometry:

Theorem 1.1. *Let M be a connected closed oriented smooth 4-manifold. Making use of its smooth structure only, out of M a von Neumann algebra $\mathfrak{R}$ can be constructed which is geometric in the sense that it is generated by geometric operators, including all complexified algebraic (i.e., formal or stemming from a metric) curvature tensors of M. Moreover $\mathfrak{R}$ itself is a hyperfinite factor of type II_1 hence is unique up to abstract isomorphisms of von Neumann algebras.*

Furthermore M admits an embedding $M \subset \mathfrak{R}$ via projections. Two 4-manifolds M,N with corresponding embeddings have abstractly isomorphic von Neumann algebras however not canonically. Nevertheless different abstract isomorphisms between these von Neumann algebras induce orientation-preserving diffeomorphisms of M and N respectively i.e., leave their embeddings unchanged. Hence up to diffeomorphisms, all connected closed oriented smooth 4-manifolds embed into a commonly given abstract von Neumann algebra $\mathfrak{R}$ which is the hyperfinite II_1 factor.

The occurrence of the hyperfinite II_1 factor in low dimensional differential topology is not only an immense source for new representations, but even brings a smooth 4-manifold invariant to life:

Theorem 1.2. *Assuming M to be as before, its von Neumann algebra $\mathfrak{R}$ admits a non-faithful representation on a certain complex separable Hilbert space, such that the unitary equivalence class of this representation is invariant under orientation-preserving diffeomorphisms of M. Consequently the Murray–von Neumann coupling constant of this representation gives rise to a smooth 4-manifold invariant q taking values in the semi-open real interval $[0,1)$. Moreover there exists a subfactor $0 \neq \mathfrak{I}(M) \subsetneqq \mathfrak{R}$ having index $[\mathfrak{R} : \mathfrak{I}(M)] \in [4,+\infty)$ such that $q(M) = 1 - \frac{4}{[\mathfrak{R}:\mathfrak{I}(M)]}$ and q behaves like $q(M\#N) = q(M) + q(N) - q(M)q(N)$ under taking connected sum.*

Let us make some general comments here. The spectrum of the possible Jones indices looks like $\left\{4\cos^2\left(\frac{\pi}{n}\right) \,\middle|\, n = 3,4,\dots\right\} \cup [4,+\infty]$ hence splits into a discrete and a continuous part. Subfactors with index in the discrete part $\{4\cos^2\left(\frac{\pi}{n}\right) \mid n \geqq 3\}$ have been completely classified [15] and in turn they follow an *ADE* pattern [14]. This classification scheme can be pushed further in an affine form to cover the case when the index is exactly 4 as well [13]. However in general the set of subfactors with index belonging to the continuous portion $[4,+\infty)$ is very wild and only partial results are known mainly for the subinterval $[4,5] \subset [4,+\infty)$ or a bit more (cf. *e.g.* [13] for an excellent survey and recent results while for some further extension cf. [1]). Concerning the impact of this division on the numerical invariant q, the previous Theorem 1.2 says that for a closed smooth 4-manifold M its invariant always satisfies $\frac{4}{1-q(M)} \in [4,+\infty)$ *i.e.*, the corresponding Jones index already belongs to the continuous range. This observation is a hint that smooth 4-manifolds might provide a rich reservoir of exotic subfactors in the wild i.e., continuous index range. In fact, as we will see here, the collection of connected closed orientable smooth 4-manifolds apparently map injectively into the collection of finite-index subfactors in the continuous range of the hyperfinite II_1 factor. Consequently the difficult task of understanding smoothness phenomena in precisely 4 dimensions seems to be related with the also difficult problem of taking an enumeration of subfactors of the hyperfinite II_1 factor in the continuous-index range. This new link with 4 dimensional differential topology should not come as a surprise in light of the existence of Jones' powerful and by-now classical polynomial knot invariant playing a role in 3-manifold theory.

Returning to the smooth invariant introduced in Theorem 1.2 it turns out that q is not really sensitive in individual cases. However, unlike other known smooth invariants which are only partially defined and take discrete, typically integer values, q is defined over a large ensemble of 4-manifolds and its image lies within a real interval hence, when considered collectively, it exhibits certain continuity properties. This allows us to make an interesting observation namely that the global behaviour of q over a hypothetical infinite sequence of smooth 4-manifolds contains information about a particular but important case, the 4-sphere. More precisely

Theorem 1.3. *Let M be a connected closed orientable smooth 4-manifold as before but having finite fundamental group. Then* $q(M) = 1 - |\pi_1(M)|\left(\frac{2}{3}\right)^{2\,\mathrm{rk}(\pi_2(M))}$.

Furthermore if there exists a sequence $\{M_n\}_{n=1,2,\dots}$ of this sort of 4-manifolds whose 1^{st} and 2^{nd} homotopy groups behave asymptotically like $|\pi_1(M_n)| \to +\infty$ and $\left|\frac{\log|\pi_1(M_n)|}{\log\frac{9}{4}} - \mathrm{rk}(\pi_2(M_n))\right| \to 0$ whenever $n \to +\infty$ then the 4 dimensional smooth Poincaré conjecture fails more precisely, the 4-sphere admits countable infinitely many inequivalent smooth structures.

Unfortunately we cannot say anything here whether or not such remarkable sequence of 4-manifolds exists. (But we expect that, knowing the difficulties surrounding the 4 dimensional smooth Poincaré conjecture, that either such sequence of 4-manifolds trivially does not exist or proving its existence is very complicated, cf. *e.g.* the survey in [6].)

The paper is organized as follows. Section 2 contains the detailed constructions leading to the proof of Theorem 1.1 while Section 3 exhibits the proofs of Theorems 1.2 and 1.3 through a sequence of separate lemmata.

2 Emergence of the hyperfinite II_1 factor

Since our earlier expositions [7, 8] are not widely-known as well as are yet technically unsatisfactory at some steps, in this section for the Reader's convenience we recall again the material in [8], however in a substantially modified and extended form, in order to prove Theorem 1.1. First we shall exhibit a simple self-contained two-step construction of a von Neumann algebra attached to any connected closed oriented smooth 4-manifold. Then the structure of this algebra will be explored in some detail.

Construction of an algebra. Take the isomorphism class of a connected closed oriented smooth 4-manifold (without boundary) and from now on let M be a once and for all fixed representative in it carrying the action of its own group of orientation-preserving diffeomorphisms $\mathrm{Diff}^+(M)$. Among all tensor bundles $T^{(p,q)}M$ over M the 2^{nd} exterior power $\wedge^2 T^*M \subset T^{(0,2)}M$ is the only one which can be endowed with a pairing in a natural way *i.e.*, a pairing extracted from the smooth structure (and the orientation) of M alone. Indeed, consider its associated vector space $\Omega^2(M) := C^\infty(M; \wedge^2 T^*M)$ of smooth 2-forms on M. Define an L^2-pairing $\langle\,\cdot\,,\,\cdot\,\rangle_{L^2(M)} : \Omega^2(M) \times \Omega^2(M) \to \mathbb{R}$ via integration:

$$\langle \varphi, \psi \rangle_{L^2(M)} := \int_M \varphi \wedge \psi \qquad (1)$$

and observe that this pairing is symmetric, non-degenerate however *indefinite* in general thus can be regarded as an indefinite L^2-scalar product on $\Omega^2(M)$. Therefore taking the complexified vector space $\Omega^2(M;\mathbb{C}) := C^\infty(M; \wedge^2 T^*M \otimes_{\mathbb{R}} \mathbb{C})$ and the $\mathbb{C}$-bilinear extension of (1), the assignment $\varphi \mapsto \langle\,\cdot\,, \varphi\rangle^{\mathbb{C}}_{L^2(M)}$ gives rise to a canonical inclusion $\Omega^2(M;\mathbb{C}) \subset (\Omega^2(M;\mathbb{C}))^*$ into the bare linear algebraic dual space. Consider $\mathrm{End}(\Omega^2(M;\mathbb{C}))$, the unital algebra of all bare $\mathbb{C}$-linear endomorphisms of $\Omega^2(M;\mathbb{C})$ and let $C(M) \subset \mathrm{End}(\Omega^2(M;\mathbb{C}))$ denote the canonical complex subalgebra generated by the embedding $\Omega^2(M;\mathbb{C}) \subset (\Omega^2(M;\mathbb{C}))^*$ *i.e.*, $C(M)$ is generated by $\Omega^2(M;\mathbb{C}) \otimes_{\mathbb{C}} \Omega^2(M;\mathbb{C})$ through the $\mathbb{C}$-linear inclusions $\Omega^2(M;\mathbb{C}) \otimes_{\mathbb{C}} \Omega^2(M;\mathbb{C}) \subset \Omega^2(M;\mathbb{C}) \otimes_{\mathbb{C}} (\Omega^2(M;\mathbb{C}))^* \subset \mathrm{End}(\Omega^2(M;\mathbb{C}))$. By definition elements of $C(M)$ look like $\omega \mapsto \sum\limits_{\ell} \langle \omega, \varphi_\ell \rangle^{\mathbb{C}}_{L^2(M)} \psi_\ell$ such that *for every* $\omega \in \Omega^2(M;\mathbb{C})$ the right hand side *exists in* $\Omega^2(M;\mathbb{C})$; in particular using any $\mathbb{C}$-linear basis $\{\varphi_1, \varphi_2, \dots\}$ satisfying $\langle \varphi_i, \varphi_j \rangle^{\mathbb{C}}_{L^2(M)} = \delta_{ij}$ then $\sum\limits_{\ell} \langle \omega, \varphi_\ell \rangle^{\mathbb{C}}_{L^2(M)} \varphi_\ell = \omega$ shows that $C(M)$ is unital. For clarity note that being (1) a non-local operation, $C(M)$ is a global infinite dimensional object, associated to the oriented smooth manifold M.

First we want to dissolve $C(M)$ into an infinite tensor product of matrix algebras. Start with an open covering of M and by compactness select a finite open subcovering $\{U_i\}_{i=1,\dots,m}$ of it; we suppose that all U_i's are already homeomorphic with open 4-balls. Take a partition of unity $\{\rho_i\}_{i=1,\dots,m}$ subordinate to the covering (*i.e.*, a collection of non-negative smooth functions satisfying $\sum\limits_{i=1}^{m} \rho_i = 1$ such that the support of ρ_i is contained in U_i). Put $\Omega^2_{\rho_i,0}(U_i) := \left\{\rho_i\psi_i \mid \psi_i \in \Omega^2(U_i;\mathbb{C}) \text{ such that } \rho_i\psi_i|_{\partial\overline{U}_i} = 0\right\}$ and writing U and ρ for any of the U_i's and their corresponding ρ_i's, let $C(U)$ be the algebra generated by $\Omega^2_{\rho,0}(U) \otimes_{\mathbb{C}} \Omega^2_{\rho,0}(U)$ as for $C(M)$ and consider $\mathfrak{M}_m\big(C(U)\big)$, the algebra of $m \times m$ matrices over $C(U)$. Picking $\rho_i\varphi_i, \rho_i\psi_i \in \Omega^2_{\rho_i,0}(U_i)$ and setting $\varphi := \sum\limits_i \rho_i\varphi_i$, $\psi := \sum\limits_i \rho_i\psi_i \in \Omega^2(M;\mathbb{C})$, the equality $\sum\limits_{i,j=1}^{m} \langle\,\cdot\,, \rho_i\varphi_i\rangle^{\mathbb{C}}_{L^2(M)} \rho_j\psi_j = \langle\,\cdot\,,\varphi\rangle^{\mathbb{C}}_{L^2(M)}\psi$ induces a homomorphism from $\mathfrak{M}_m(C(U))$ onto $C(M)$. This homomorphism is surjective however not injective: both follow from the associated decomposition $\varphi = \sum\limits_i (\rho_i\varphi) \mapsto \bigoplus\limits_i (\rho_i\varphi)$ which is not unique due to overlappings in the open covering *i.e.*, because the open subset of intersections

$$\bigcup_{i,j=1}^{m} U_i \cap U_j \subset M \tag{2}$$

is not empty. Consequently this surjective homomorphism has a non-trivial kernel $J(U) \subset \mathfrak{M}_m(C(U))$ yielding an isomorphism $C(M) \cong \mathfrak{M}_m(C(U))/J(U)$. Proceeding further, let us say that the open covering $\{V_k\}_{k=1,\dots,mn}$ of M is an *n-homogeneous refinement* of $\{U_i\}_{i=1,\dots,m}$ if for every index $1 \leqq k \leqq mn$ there exists $1 \leqq i \leqq m$ satisfying $V_k \subset U_i$ and every U_i contains precisely n number of V_k's which are again (smaller) open 4-balls. Then repeating the previous decomposition procedure we obtain further that $C(M) \cong \mathfrak{M}_{mn}(C(V))/J(V) = \mathfrak{M}_m\big(\mathfrak{M}_n\big(C(V)\big)\big)/J(V)$ where $J(V) \subset \mathfrak{M}_{mn}(C(V))$ is the ideal generated by the overlappings in the refined covering $\{V_k\}_{k=1,\dots,mn}$ as before. By its definition as a topological space, M admits a countable basis and any particular countable sequence of nested open subsets $M \supset U \supset V \supset \dots$ taken from the above refinement respectively, possesses the Cantor property *i.e.*, has a non-empty intersection; we can suppose without loosing generality that it is the particular point $x \in M$, and note that every point of M arises in this manner. Therefore the induced sequence $C(M) \supset C(U) \supset C(V) \supset \dots$ of nested local unital algebras terminates at the corresponding full infinitesimal algebra $C(x) = \mathrm{End}(\wedge^2 T^*_x M \otimes_{\mathbb{R}} \mathbb{C}) \cong \mathfrak{M}_6(\mathbb{C})$ since $\dim_{\mathbb{C}}(\wedge^2 T^*_x M \otimes_{\mathbb{R}} \mathbb{C}) = \binom{4}{2} = 6$. The corresponding sequence of ideals $0 = J(M), J(U), J(V), \dots$ likewise terminates at $J(x)$. Geometrically speaking, since the points of M are already disjoint hence the open subset (2) of overlappings tends to the empty set during refinements, we get $J(x) = 0$. Algebraically, since the ideals of $\mathfrak{M}_k(R)$ are in one one-to-one correspondence with the ideals of R we can identify the ideals above with those within $C(M), C(U), C(V), \dots$ respectively. However since $C(x) \cong \mathfrak{M}_6(\mathbb{C})$ has no non-trivial ideals, we find $J(x) = 0$. Consequently the ω-completed (in the sense of set theory) step-by-step procedure of taking homogeneous refinements of a finite open covering of M formally induces a countable infinite resolution $C(M)\cong\mathfrak{M}_m\big(\mathfrak{M}_n(\dots(C(x))\dots)\big)/J(x)\cong\mathfrak{M}_m\big(\mathfrak{M}_n(\dots(\mathfrak{M}_6(\mathbb{C}))\dots)\big)\cong\mathfrak{M}_{mn\dots}(\mathbb{C})$. This procedure can be captured precisely by the concept of an infinite tensor product: $C(M)$ arises as the inductive limit of matrix algebras $C_n(M)=\mathfrak{M}_{k_1\dots k_n}(\mathbb{C})=\mathfrak{M}_{k_1}(\mathbb{C})\otimes_{\mathbb{C}}\cdots\otimes_{\mathbb{C}}\mathfrak{M}_{k_n}(\mathbb{C})$ where the embedding $C_n(M)\subset C_{n+1}(M)$ is by means of the mapping $A\mapsto A\otimes 1$ where $1\in\mathfrak{M}_{k_{n+1}}(\mathbb{C})$ is the unit. To summarize: there exists a decomposition

$$C(M)\cong\mathfrak{M}_{k_1}(\mathbb{C})\otimes_{\mathbb{C}}\mathfrak{M}_{k_2}(\mathbb{C})\otimes_{\mathbb{C}}\dots \tag{3}$$

yet depending on a particular step-by-step homogeneous refinement procedure of a particular finite covering of M; nevertheless by its construction $C(M)$ contains the infinitesimal hence finite dimensional algebras $\mathrm{End}(\wedge^2T^*_xM\otimes_{\mathbb{R}}\mathbb{C})$ at every $x\in M$ as well as forms part of the global algebra $\mathrm{End}(\Omega^2(M;\mathbb{C}))$ yielding canonical inclusions

$$C^\infty(M;\mathrm{End}(\wedge^2T^*M\otimes_{\mathbb{R}}\mathbb{C}))\subset C(M)\subset\mathrm{End}(\Omega^2(M;\mathbb{C}))=\mathrm{End}\left(C^\infty(M;\wedge^2T^*M\otimes_{\mathbb{R}}\mathbb{C})\right) \tag{4}$$

of $\mathbb{C}$-algebras by construction.

Next we carry out completion. We see via (3) that $C(M)$ is a complex $\circledast$-algebra whose $\circledast$-operation (provided by taking Hermitian matrix transpose, a non-local operation) is written as $A\mapsto A^\circledast$. The isomorphism (3) also shows that if $A\in C(M)$ then one can pick the smallest $n\in\mathbb{N}$ such that $A\in C_n(M)$, consequently A has a finite trace defined by $\tau(A):=\frac{1}{k_1k_2\dots k_n}\mathrm{Trace}(A)$ *i.e.*, taking the usual normalized trace of the corresponding $(k_1\dots k_n)\times(k_1\dots k_n)$ complex matrix. Observe that $\tau(A)\in\mathbb{C}$ does not depend on n. (We will see shortly that diffeomorphisms act by inner automorphisms on $C(M)$ hence by [18, Corollary 2] also $\tau: C(M)\to\mathbb{C}$ is the only state which is $\mathrm{Diff}^+(M)$-invariant *i.e.*, natural to use in our situation.) Following [2, Section 1.6] we then define a sesquilinear inner product on $C(M)$ by $(A,B):=\tau(AB^\circledast)$ which is non-negative and non-degenerate thus the completion of $C(M)$ with respect to the norm $\|\cdot\|$ induced by $(\,\cdot\,,\,\cdot\,)$ renders $C(M)$ a complex Hilbert space what we shall write as $\mathscr{H}$ and its Banach algebra of all bounded linear operators as $\mathfrak{B}(\mathscr{H})$. Multiplication in $C(M)$ from the left on itself is continuous hence gives rise to a representation $\pi: C(M)\to\mathfrak{B}(\mathscr{H})$. Finally our key object effortlessly emerges as the weak closure of the image of $C(M)$ under π within $\mathfrak{B}(\mathscr{H})$ or equivalently, by referring to von Neumann's bicommutant theorem [2, Theorem 2.1.3] we put

$$\mathfrak{R}:=(\pi(C(M)))''\subset\mathfrak{B}(\mathscr{H})\ .$$

This von Neumann algebra of course admits a unit $1\in\mathfrak{R}$ moreover continues to have trivial center *i.e.*, is a factor. Moreover by construction it is hyperfinite. The trace τ as defined extends from $C(M)$ to $\mathfrak{R}$ and satisfies $\tau(1)=1$. Moreover [2, Proposition 4.1.4] this trace is unique on $\mathfrak{R}$. Consequently $\mathfrak{R}$ is a hyperfinite II_1 factor hence is unique up to abstract isomorphisms [2, Theorem 11.2.2]. In particular in this sense its construction is independent of the involved covering-refinement procedure over M. Going further, we also obtain by extension a representation $\pi:\mathfrak{R}\to\mathfrak{B}(\mathscr{H})$. Observe that here we have constructed $\mathscr{H}$ as a completion of $(C(M),\tau)$ however the same $\mathscr{H}$ arises if taking the completion of $(\mathfrak{R},\tau)$. Hence the two kinds of completions $\mathfrak{R}$ and $\mathscr{H}$ of one and the same object $C(M)$ in fact form an increasing chain $C(M)\subset\mathfrak{R}\subset\mathscr{H}$ as complex (complete) vector spaces. Given $A,B\in\mathfrak{R}$ we shall write $A\in\mathfrak{R}$ but $\hat{B}\in\mathscr{H}$ from now on as usual. This is necessary since $\mathfrak{R}$ and $\mathscr{H}$ are very different for example as $\mathrm{U}(\mathscr{H})$-modules: given a unitary operator $V\in\mathrm{U}(\mathscr{H})$ then $A\in\mathfrak{R}$ is acted upon as $A\mapsto VAV^{-1}$ but $\hat{B}\in\mathscr{H}$ transforms as $\hat{B}\mapsto V\hat{B}$. Using this notation the trace always can be written as a scalar product with the image of the unit in $\mathscr{H}$ that is, for every $A\in\mathfrak{R}$ we have $\tau(A)=(\hat{A},\hat{1})$ exhibiting a general abstract expression for the trace.

Exploring the algebra $\mathfrak{R}$. Before proceeding further let us make a digression here to gain a better picture. This is desirable because taking the weak closure like $\mathfrak{R}$ of some explicitly known structure like $C(M)$, often involves a sort of loosing control over the latter. Nevertheless we already know promisingly that $\mathfrak{R}$ is a hyperfinite factor von Neumann algebra of II_1 type. Let us now reveal some of its elements.

1. We immediately recognize via the left hand side of (4) combined with the canonical inclusion $C(M) \subset \mathfrak{R}$ that $C^\infty(M;\mathrm{End}(\wedge^2 T^*M \otimes_{\mathbb{R}} \mathbb{C})) \subset \mathfrak{R}$ *i.e.*, the bundle or in other words pointwisely defined endomorphisms of 2-forms form a part of $\mathfrak{R}$. As an introductory simple observation note that in case of such elements all operations in $\mathfrak{R}$ stem from the corresponding pointwise operations: if $a, b \in \mathbb{C}$ and $A, B \in C^\infty(M;\mathrm{End}(\wedge^2 T^*M \otimes_{\mathbb{R}} \mathbb{C}))$ therefore $A_x, A_x^*, B_x \in \mathrm{End}(\wedge^2 T_x^*M \otimes_{\mathbb{R}} \mathbb{C})$ for every $x \in M$ then readily $(aA + bB)_x = aA_x + bB_x$, $(AB)_x = A_x B_x$ and $A_x^{\circledast} = A_x^*$. Operators of this kind are important because they allow to make a contact with local *four dimensional* differential geometry.[1] A peculiarity of four dimensions is that the space $C^\infty(M;\mathrm{End}(\wedge^2 T^*M \otimes_{\mathbb{R}} \mathbb{C}))$ contains curvature tensors (more precisely their complex linear extensions) on M. If (M, g) is an oriented Riemannian 4-manifold then its Riemannian curvature tensor R_g is indeed a member of this subalgebra: with respect to the splitting of complexified 2-forms into their (anti)self-dual parts its complex linear extension looks like (cf. [17])

$$R_g = \begin{pmatrix} \frac{1}{12}\mathrm{Scal} + \mathrm{Weyl}^+ & \mathrm{Ric}_0 \\ \mathrm{Ric}_0^* & \frac{1}{12}\mathrm{Scal} + \mathrm{Weyl}^- \end{pmatrix} \quad : \quad \begin{matrix} \Omega^+(M;\mathbb{C}) \\ \bigoplus \\ \Omega^-(M;\mathbb{C}) \end{matrix} \longrightarrow \begin{matrix} \Omega^+(M;\mathbb{C}) \\ \bigoplus \\ \Omega^-(M;\mathbb{C}) \end{matrix} \tag{5}$$

hence is a self-adjoint operator $R_g^{\circledast} = R_g$. More generally, $C^\infty(M;\mathrm{End}(\wedge^2 T^*M \otimes_{\mathbb{R}} \mathbb{C}))$ hence $\mathfrak{R}$ contains the complex linear extensions of all algebraic (*i.e.*, formal only, not stemming from a metric) curvature tensors R over M.

Whenever $A \in C^\infty(M;\mathrm{End}(\wedge^2 T^*M \otimes_{\mathbb{R}} \mathbb{C}))$ one can work out an explicit formula for its trace because one can compare the global trace $\tau(A)$ and the local trace function $x \mapsto \mathrm{tr}(A_x)$ given by the pointwise traces of the local operators $A_x : \wedge^2 T_x^*M \otimes_{\mathbb{R}} \mathbb{C} \to \wedge^2 T_x^*M \otimes_{\mathbb{R}} \mathbb{C}$ at every $x \in M$. Recall that $\mathfrak{R}$ has been constructed as the weak closure of the algebra (3). In fact [2, Section 1.1.6] the universality of $\mathfrak{R}$ permits to obtain it from any particular sequence of matrix algebras, like *e.g.* $\mathfrak{M}_6(\mathbb{C}) \otimes_{\mathbb{C}} \mathfrak{M}_6(\mathbb{C}) \otimes_{\mathbb{C}} \ldots$ whose weak closure therefore is again $\mathfrak{R}$. Choose an arbitrary Riemannian metric g on the closed oriented M, extend it sesquilinearly and take the corresponding definite L^2-scalar product on $\Omega^2(M;\mathbb{C})$. That is, put

$$(\varphi, \psi)^{\mathbb{C}}_{L^2(M,g)} := \int_M \varphi \wedge *_g \overline{\psi} = \int_{x \in M} g_x(\varphi_x, \psi_x)^{\mathbb{C}} \mu_g(x)$$

as usual, where $\mu_g \in \Omega^4(M)$ is the induced volume form on (M, g). Pick an orthonormal frame $\{\varphi_1, \varphi_2, \ldots\}$ in $\Omega^2(M;\mathbb{C})$. By the aid of these tools the trace of A takes the shape

$$\tau(A) = \lim_{n \to +\infty} \frac{1}{6^n} \sum_{i=1}^{6^n} (A\varphi_i\,,\, \varphi_i)^{\mathbb{C}}_{L^2(M,g)}\ .$$

Fix $n \in \mathbb{N}$, write $M_n := \bigcap_{i=1}^{6^n} \mathrm{supp}\varphi_i \subsetneqq M$ and take a point $x \in M_n$. Since $\dim_{\mathbb{C}}(\wedge^2 T_x^*M \otimes_{\mathbb{R}} \mathbb{C}) = \binom{4}{2} = 6$ the maximal number of completely disjoint linearly independent sub-6-tuples in $\{\varphi_{1,x}, \varphi_{2,x}, \ldots, \varphi_{6^n,x}\}$ is equal to $\frac{6^n}{6} = 6^{n-1}$. Moreover it follows from Sard's lemma that with a generic smooth choice for $\{\varphi_1, \varphi_2, \ldots\}$ the subset of those points $y \in M_n$ where this number is less than 6^{n-1} has measure zero in

[1] In fact all the constructions so far work for an arbitrary closed oriented smooth $4k$-manifold with $k = 0, 1, 2, \ldots$ (note that in $4k+2$ dimensions the pairing (1) gives rise to a symplectic structure on $2k+1$-forms).

M_n with respect to the measure μ_g. Consequently

$$\sum_{i=1}^{6^n}(A\varphi_i,\varphi_i)^{\mathbb{C}}_{L^2(M_n,g)} = \int\limits_{x\in M_n}\sum_{i=1}^{6^n} g_x(A_x\varphi_{i,x},\varphi_{i,x})^{\mathbb{C}}\,\mu_g(x) = 6^{n-1}\int\limits_{x\in M_n}\mathrm{tr}(A_x)\mu_g(x)\ .$$

Since $\{\varphi_1,\varphi_2,\dots\}$ is a basis in $\Omega^2(M;\mathbb{C})$ therefore $M\setminus\bigcup\limits_{n=0}^{+\infty}M_n = \bigcap\limits_{n=0}^{+\infty}(M\setminus M_n)$ has measure zero as well we can let $n\to+\infty$ to end up with

$$\tau(A) = \frac{1}{6}\int\limits_M \mathrm{tr}(A)\mu_g\ .$$

Notice by inserting (5) that $\tau(R_g) = \frac{1}{12}\int_M \mathrm{Scal}\,\mu_g$ hence the operator algebraic trace on the associated von Neumann algebra is nothing else than the generalization of the total scalar curvature in Riemannian geometry (or the Einstein–Hilbert action of vacuum general relativity).

2. Next we assert that the group of invertible elements within $C(M)$ can be canonically identified with $\mathrm{Iso}(\wedge^2T^*M\otimes_{\mathbb{R}}\mathbb{C})$, the group of bundle isomorphisms *i.e.*, pairs (Φ,F) consisting of an orientation-preserving diffeomorphism Φ of the base M and a fiberwise $\mathbb{C}$-linear diffeomorphism F of the total space $\wedge^2T^*M\otimes_{\mathbb{R}}\mathbb{C}$ such that

$$\begin{array}{ccc} \wedge^2T^*M\otimes_{\mathbb{R}}\mathbb{C} & \xrightarrow{F} & \wedge^2T^*M\otimes_{\mathbb{R}}\mathbb{C}\\ \pi\big\downarrow & & \big\downarrow\pi\\ M & \xrightarrow{\quad\Phi\quad} & M \end{array}$$

is commutative. On the one hand by the aid of the right hand side inclusion in (4) invertible elements in $C(M)$ belong to $\mathrm{End}(\Omega^2(M;\mathbb{C}))$; on the other hand it is clear that smooth bundle isomorphisms act on the space of smooth sections in an obvious $\mathbb{C}$-linear way hence $\mathrm{Iso}(\wedge^2T^*M\otimes_{\mathbb{R}}\mathbb{C})\subset\mathrm{End}(\Omega^2(M;\mathbb{C}))$ too. Consequently our assertion at least makes sense. Moreover intuitively it also follows at once, for the very geometric content of the decomposition (3) in its ω-completed form is that elements of $C(M)$, while operating on smooth sections, preserve their infinitesimal structure *i.e.*, the individual fibers of $\wedge^2T^*M\otimes_{\mathbb{R}}\mathbb{C}$ too, hence the invertible ones give rise to bundle morphisms.

More precisely, by recalling again the construction leading to the decomposition (3), take a finite open covering $\{U_i\}_{i=1,\dots,m}$ of M, pick an invertible operator $A\in C(M)=\mathfrak{M}_m(C(U))/J(U)$ and consider any of its preimage $A_m\in\mathfrak{M}_m(C(U))$. This is an $m\times m$ invertible matrix consequently admits a unique "LU-decomposition" *i.e.*, $A_m = T'_m\Pi_mT''_m$ such that $T'_m,T''_m\in\mathfrak{M}_m(C(U))$ are upper triangular matrices moreover $\Pi_m\in\mathfrak{M}_m(C(U))$ is a permutation matrix (this is meaningful here because $0,1\in C(U)$ always); let π_m be the corresponding permutation of the index set $\{1,2,\dots,m\}$. In this way, on the one hand, A_m induces an approximate diffeomorphism of $\{U_i\}_{i=1,\dots,m}$ by putting $\Phi_m(U_i):=U_{\pi_m(i)}$ for every $i=1,\dots,m$. On the other hand, A_m gives rise to an approximate element F_m (by considering its action on very localized 2-forms), which is compatible with Φ_m. These together comprise an approximating bundle isomorphism (Φ_m,F_m). Then taking a countable sequence of step-by-step homogeneous refinements as before we obtain a bundle isomorphism (Φ,F) assigned to A. Conversely, given a pair (Ψ,G), on the one hand take an open covering $\{V_j\}_{j=1,\dots,n}$ which is invariant under Ψ. On the other hand viewing G as an invertible $\mathbb{C}$-linear transformation of $\Omega^2(M;\mathbb{C})$ consider the matrix $B_n\in\mathfrak{M}_n(C(V))$ built up from the action of G using localized 2-forms belonging to $\Omega^2_{\rho_j,0}(V_j)$ for every $j=1,\dots,n$ that is, define $(B_n)_{ij}$ by solving $G=\langle\,\cdot\,,\rho_i\psi_i\rangle^{\mathbb{C}}_{L^2(M)}\rho_j\psi_j$ with suitable local 2-forms, and take the image of this matrix in $C(M)=\mathfrak{M}_n(C(V))/J(V)$. Then carrying out refinements as usual we obtain an invertible operator $B\in C(M)$.

Note that the identification of invertible elements of $C(M)$ with bundle isomorphisms has already been partly recognized in the previous item, for the left hand side embedding of (4) implies that the gauge group $\mathscr{G}(\wedge^2T^*M\otimes_{\mathbb{R}}\mathbb{C})=C^\infty(M;\mathrm{Aut}(\wedge^2T^*M\otimes_{\mathbb{R}}\mathbb{C}))$ of $\wedge^2T^*M\otimes_{\mathbb{R}}\mathbb{C}$ embeds into the invertible part of $C(M)$. Indeed, there exists a short exact sequence

$$1\longrightarrow\mathscr{G}(\wedge^2T^*M\otimes_{\mathbb{R}}\mathbb{C})\longrightarrow\mathrm{Iso}(\wedge^2T^*M\otimes_{\mathbb{R}}\mathbb{C})\longrightarrow\mathrm{Diff}^+(M)\longrightarrow 1$$

which injects the gauge group into the isomorphism group. Moreover $\Phi\mapsto(\Phi^{-1})^*$ as a bundle pullback gives a group injection $\mathrm{Diff}^+(M)\subset\mathrm{Iso}(\wedge^2T^*M\otimes_{\mathbb{R}}\mathbb{C})$ too whose composition with the projection $\mathrm{Iso}(\wedge^2T^*M\otimes_{\mathbb{R}}\mathbb{C})\to\mathrm{Diff}^+(M)$ is the identity. Consequently this sequence can be supplemented to

$$1\longrightarrow\mathscr{G}(\wedge^2T^*M\otimes_{\mathbb{R}}\mathbb{C})\longrightarrow\mathrm{Iso}(\wedge^2T^*M\otimes_{\mathbb{R}}\mathbb{C})\rightleftarrows\mathrm{Diff}^+(M)\longrightarrow 1 \tag{6}$$

implying a splitting $\mathrm{Iso}(\wedge^2T^*M\otimes_{\mathbb{R}}\mathbb{C}))=\mathscr{G}(\wedge^2T^*M\otimes_{\mathbb{R}}\mathbb{C})\rtimes\mathrm{Diff}^+(M)$ as a semi-direct product. Therefore $\mathrm{Diff}^+(M)\subset C(M)$ too, and $\Phi\in\mathrm{Diff}^+(M)$ acts on $\Omega^2(M;\mathbb{C})$ by the corresponding pull-back $(\Phi^{-1})^*$ on 2-forms which is a $\mathbb{C}$-linear operator. Composing this with $C(M)\subset\mathfrak{R}$ we find that $\mathrm{Diff}^+(M)\subset\mathfrak{R}$ *i.e.*, the group of orientation-preserving diffeomorphisms form a part of $\mathfrak{R}$ too.

Furthermore, it is easy to see that $\mathrm{Diff}^+(M)\subset\mathfrak{R}$ acts by unitaries on $\mathscr{H}$ via the standard representation $\pi:\mathfrak{R}\to\mathfrak{B}(\mathscr{H})$. Using any definite L^2-scalar product on $\Omega^2(M;\mathbb{C})$ and a corresponding base $\{\varphi_1,\varphi_2,\dots\}$ as above, we can represent $(\Phi^{-1})^*$ by a countable infinite complex matrix according to (3) whose Hermitian conjugate $((\Phi^{-1})^*)^{\circledast}$ satisfies $\big(\varphi,(\Phi^{-1})^*\psi\big)_{L^2(M,g)}=\big(((\Phi^{-1})^*)^{\circledast}\varphi,\psi\big)_{L^2(M,g)}$. But the diffeomorphism-invariance of the L^2-scalar product implies

$$\big(((\Phi^{-1})^*)^{\circledast}\varphi,\psi\big)^{\mathbb{C}}_{L^2(M,g)}=\big(\varphi,(\Phi^{-1})^*\psi\big)^{\mathbb{C}}_{L^2(M,g)}=\big(\Phi^*\varphi,\Phi^*(\Phi^{-1})^*\psi\big)^{\mathbb{C}}_{L^2(M,g)}=\big(\Phi^*\varphi,\psi\big)^{\mathbb{C}}_{L^2(M,g)}$$

demonstrating $((\Phi^{-1})^*)^{\circledast}=\Phi^*$ that is, diffeomorphisms are unitary over $\big(\Omega^2(M;\mathbb{C})\,,(\,\cdot\,,\,\cdot\,)_{L^2(M,g)}\big)$. Consider now the dense subset of $\mathscr{H}$ having elements of the form $\hat{A}\in\mathscr{H}$ where $A\in C(M)\subset\mathfrak{R}$. Then taking the scalar product $(\,\cdot\,,\,\cdot\,)$ of $\mathscr{H}$ and exploiting the cyclic property of the trace as well as the already recognized unitarity of Φ on $\big(\Omega^2(M;\mathbb{C})\,,(\,\cdot\,,\,\cdot\,)_{L^2(M,g)}\big)$ we find that

$$\begin{aligned}\big(\pi((\Phi^{-1})^*)\hat{A},\pi((\Phi^{-1})^*)\hat{B}\big)\;&=\;\tau\big((\Phi^{-1})^*A((\Phi^{-1})^*B)^{\circledast}\big)=\tau\big(((\Phi^{-1})^*)^{\circledast}(\Phi^{-1})^*AB^{\circledast}\big)=\tau(AB^{\circledast})\\&=\;\big(\hat{A},\hat{B}\big)\end{aligned}$$

proving that the embedding $\mathrm{Diff}^+(M)\subset\mathfrak{R}$ provided by $\Phi\mapsto(\Phi^{-1})^*$ is indeed unitary. This also demonstrates in particular that $|\tau((\Phi^{-1})^*)|=1$.

3. Last but not least we exhibit an especially important class of elements in $\mathfrak{R}$ which permit to inject M back into its $\mathfrak{R}$ as follows. First write $M':=M$ and put $P_{M'}:=0\in C(M)=\mathfrak{M}_1(C(M))$. Next, take a finite open covering $\{U_i\}_{i=1,\dots,m}$ of M and its associated amplified algebra $\mathfrak{M}_m(C(U))$ as before. After re-indexing if necessary, consider the open subsets $U:=U_1$ and $U':=U_1\cup U_2\cup\dots\cup U_{m_1}$ with $1\leqq m_1\leqq m$ such that $U_1\cap U_i\neq\emptyset$ whenever $1\leqq i\leqq m_1$ (*i.e.*, take all subsets U_i having non-empty intersections with the distinguished open subset U_1 in the covering). Note that $U\Subset U'$ (*i.e.*, $\overline{U}\subset U'$). Picking the local zero and unit $0,1\in C(U)$ consider the diagonal matrix $P_{U'}\in\mathfrak{M}_m(C(U))$ containing 0's in the first m_1 entries and 1's in the last $m-m_1$ entries along its diagonal. This matrix maps onto a non-zero element in $C(M)\cong\mathfrak{M}_m(C(U))/J(U)$ what we continue to write as $P_{U'}$. As an operator on smooth 2-forms, it acts like $P_{U'}\varphi=P_{U'}\Big(\sum\limits_{i=1}^{m}\rho_i\varphi\Big)=\sum\limits_{i=m_1+1}^{m}\rho_i\varphi$ hence is a projection onto the subspace $\Omega^2(M,\overline{U'};\mathbb{C})\subset\Omega^2(M;\mathbb{C})$ of 2-forms vanishing on the closed subset $\overline{U'}\subsetneqq M$. Note

however that, considered as $P_{U'}\in\mathfrak{R}$ via $C(M)\subset\mathfrak{R}$, this operator is already an abstract projection too, yet characterized by its geometric behaviour that preserves $\Omega^2(M,\overline{U'};\mathbb{C})$. Conversely, in fact any projection in $\mathfrak{R}$ possessing this behaviour can be taken to play the role of a $P_{U'}$ here. Repeating the construction for a countable sequence of nested open subsets and their surrounders

$$\begin{array}{ccccccc} M & \supset & U & \supset & V & \supset & \dots \\ \| & & \cap & & \cap & & \\ M' & \supset & U' & \supset & V' & \supset & \dots \end{array}$$

collapsing to some point $x\in M$ during a countable step-by-step homogeneous refinement of the open covering as before, and referring to the decomposition (3) as well as using $C(M)\subset\mathfrak{R}$, in this way we come up with an element $P_x\in\mathfrak{R}$. This is an abstract projection of $\mathfrak{R}$ however note that P_x inherits the geometric property that it acts as the identity precisely on $\Omega^2(M,x;\mathbb{C})$ as well. Hence surely $0\neq P_x\neq 1$ and $P_x\neq P_y$ for every $x,y\in M$ such that $x\neq y$. We obtain therefore a projection $P_x\in\mathfrak{R}$ assigned to the point $x\in M$ and the resulting map

$$i_M : M\longrightarrow\mathfrak{R} \tag{7}$$

defined by $x\mapsto P_x$ is obviously injective.

If $x,y\in M$ take a smooth 1-parameter subgroup $\{\Phi_t\}_{t\in[-1,+1]}\subset\mathrm{Diff}^+(M)$ such that the curve $t\mapsto\Phi_t(x)$ connects x with y *i.e.*, $\Phi_0(x)=x$ and $\Phi_1(x)=y$ or equivalently $\Phi_1^{-1}(y)=\Phi_{-1}(y)=x$ consequently for the corresponding subspaces $(\Phi_{-1})^*\Omega^2(M,x;\mathbb{C})=\Omega^2(M,y;\mathbb{C})$. Consider the induced family of projections $P(t):=\Phi^*_{-t}P_x\Phi^*_t$ within $\mathfrak{R}$. Then $x\mapsto P_x$ is continuous in the sense that $P(0)=P_x$ and $P(1)=P_y$. In particular all of these projections are unitary equivalent hence $0<\tau(P_x)=\tau(P_y)<1$ for every $x,y\in M$. It also follows that the image of M in $\mathfrak{R}$ is always closed in any usual topology on $\mathfrak{R}$. Indeed, let $x_1,x_2,\dots$ be a countable sequence of points in M such that the corresponding series of projections $P_{x_1},P_{x_2},\dots$ in $\mathfrak{R}$ converges to a unique operator $A\in\mathfrak{R}$. Then there exist (non-unique) elements $\Phi_1,\Phi_2,\dots$ in $\mathrm{Diff}^+(M)$ such that $x_i=\Phi_i(x_1)$ hence $P_{x_i}=(\Phi_i^{-1})^*P_{x_1}\Phi_i^*$. By compactness of M there is a (sub)sequence converging to a point $x\in M$ as $j\to+\infty$ and a (non-unique) transformation Φ such that $\Phi(x_1)=x$. We can suppose that $\Phi_{x_j}\to\Phi$ in the induced subspace topology on $\mathrm{Diff}^+(M)\subset\mathfrak{R}$. Regarding the projections $P_{x_j}=(\Phi_j^{-1})^*P_{x_1}\Phi_j^*\to(\Phi^{-1})^*P_{x_1}\Phi^*=P_x$ as $j\to+\infty$ thus the original limit operator satisfies that $A=P_x$ with some unique $x\in M$. We conclude that (7) in any usual topology on $\mathfrak{R}$ gives rise to a closed embedding of M into $\mathfrak{R}$ via projections. It is probably worth recalling here the embedding of compact Riemannian spaces into Hilbert spaces via heat kernel techniques [3].

4. Having understood (7) for a given 4-manifold, let us compare them for different spaces. So let M,N be two connected closed oriented smooth 4-manifolds and consider their corresponding embeddings $M\subset\mathfrak{R}$ and $N\subset\mathfrak{R}$ into their von Neumann algebras via (7) respectively. Regardless what M or N are, their abstractly given algebras are both hyperfinite factors of II_1 type, therefore these latter objects are isomorphic [2, Theorem 11.2.2] however not in a canonical fashion. Indeed, if $F':\mathfrak{R}\to\mathfrak{R}$ is an abstract isomorphism between the $\mathfrak{R}$'s for M and N respectively, then any other abstract isomorphism between them has the form $F''=\beta^{-1}F'\alpha$ where α and β are $\mathbb{C}$-linear $*$-automorphisms (in short from now on: automorphisms) of the abstractly given $\mathfrak{R}$ for M and the abstractly given $\mathfrak{R}$ for N respectively.

Therefore to understand the freedom how operator algebras for different 4-manifolds are identified we have to understand automorphisms of $\mathfrak{R}$. First, let us see how inner automorphisms of the weakly dense subalgebra $C(M)$, which are of course conjugations with its unitary elements, look like. It follows from the splitting (6) of the group of invertible elements that an inner automorphism of $C(M)$ admits a unique decomposition $\mathrm{Ad}_\gamma\mathrm{Ad}_{(\Phi^{-1})^*}$ where $\gamma\in\mathscr{G}(\wedge^2T^*M\otimes_{\mathbb{R}}\mathbb{C})$ is a $\mathbb{C}$-linear gauge transformation of the bundle $\wedge^2T^*M\otimes_{\mathbb{R}}\mathbb{C}$ hence leaves M pointwise fixed, and $\Phi\in\mathrm{Diff}^+(M)$ is a diffeomorphism of M which also preserves M as a whole. It then readily follows that the image of M within $\mathfrak{R}$ given by (7) is preserved by this automorphism. Next, consider the strong*-topology on $\mathfrak{B}(\mathscr{H})$. Taking $\mathfrak{R} \subset \mathfrak{B}(\mathscr{H})$ it is clear that $C(M) \subset \mathfrak{R}$ is a dense subalgebra; moreover also taking the unique unitary implementation $\mathrm{Aut}\,\mathfrak{R} \subset \mathfrak{B}(\mathscr{H})$ of the full automorphism group [2, Proposition 7.5.2], it is known [16, Theorem 4] that the subgroup of inner automorphisms is also dense within $\mathrm{Aut}\,\mathfrak{R}$. Consequently $\mathrm{Inn}\,C(M)$ itself is strong*-dense in $\mathrm{Aut}\,\mathfrak{R}$ hence any automorphism α of $\mathfrak{R}$ arises as

$$\alpha = \lim_i \left(\mathrm{Ad}_{\gamma_i}\mathrm{Ad}_{(\Phi_i^{-1})^*}\right) \tag{8}$$

where the limit is taken in the strong*-topology on $\mathfrak{B}(\mathscr{H})$. Therefore, taking into account as well that M is a strong*-closed subset of $\mathfrak{R}$ under (7), we conclude that α also preserves $M \subset \mathfrak{R}$. Likewise, $\mathrm{Ad}_\delta \mathrm{Ad}_{(\Psi^{-1})^*}$ is the shape of an inner automorphism of $C(N)$ and a generic automorphism β arises as strong*-limit of them. These make sure that given an abstract isomorphism $F' : \mathfrak{R} \to \mathfrak{R}$ between the von Neumann algebras constructed for M and N respectively, then any other abstract isomorphism can be expressed as $F'' = \lim\limits_i \left(\mathrm{Ad}_{\Psi_i^*}(\mathrm{Ad}_{\delta_i^{-1}}F'\mathrm{Ad}_{\gamma_i})\mathrm{Ad}_{(\Phi_i^{-1})^*}\right)$ between them. Consequently abstract isomorphisms between a pair of abstractly given von Neumann algebras differ only by automorphisms which preserve their underlying 4-manifolds as embedded within their algebras via (7) respectively; that is, differences between identifications are inessential in this sense. Our overall conclusion here therefore is that *up to diffeomorphisms every connected closed oriented smooth* 4*-manifold M admits a closed embedding into a commonly given abstract von Neumann algebra* $\mathfrak{R}$ *via (7).*

5. We close the partial comprehension of $\mathfrak{R}$ with an observation regarding its general structure. Recall that the group of invertible elements of $C(M)$ decomposes as the semi-direct product of the group of gauge transformations of $\wedge^2 T^*M \otimes_{\mathbb{R}} \mathbb{C}$ and the group of diffeomorphisms of M via (6); thus by the inclusion $C(M) \subset \mathfrak{R}$ a weakly dense subset of $\mathfrak{R}$ therefore admits a decomposition into endomorphisms of the bundle $\wedge^2 T^*M \otimes_{\mathbb{R}} \mathbb{C}$ and diffeomorphisms of M. Consequently given a 4-manifold M, its associated von Neumann algebra $\mathfrak{R}$ is geometric in the sense that it is generated by operators having finite dimensional or geometric origin.

Summing up our findings so far. Given a connected closed oriented smooth 4-manifold M there exists a hyperfinite factor von Neumann algebra of II_1 type $\mathfrak{R}$ associated to M such that the solely input in its construction has been the pairing (1). Hence $\mathfrak{R}$ depends only on the orientation and the smooth structure of M. It contains, certainly among many abstract operators, the space M itself as projections, its orientation-preserving diffeomorphisms moreover, as a fingerprint of four dimensions, the space of algebraic curvature tensors. Nevertheless $\mathfrak{R}$ is yet geometric in the sense that it is generated by M's local operators alone. It is remarkable that despite the *plethora* of smooth 4-manifolds detected since the early 1980's their associated von Neumann algebras here are unique offering a sort of justification terming $\mathfrak{R}$ as "universal". Moreover one is permitted to say that every connected oriented smooth 4-manifold M (perhaps together with its curvature tensor) embeds up to diffeomorphisms hence in a functorial way into a *common* $\mathfrak{R}$, and to look upon this von Neumann algebra as a natural common non-commutative space generalization, in the spirit of non-commutative geometry [5], of all oriented smooth 4-manifolds (or all 4-geometries). This universality also justifies the simple notation $\mathfrak{R}$ used throughout the text.

Proof of Theorem 1.1: Putting together all considerations so far the theorem is proved. $\square$

3 Some representations of the hyperfinite II_1 factor

Representations of $\mathfrak{R}$ and a new smooth 4-manifold invariant. We can now proceed further: the next lemmata closely follow [8, Lemmata 2.1-2.4] but with substantially improved constructions and extended contents.

Lemma 3.1. *Let M be a connected closed oriented smooth 4-manifold and $\mathfrak{R}$ its von Neumann algebra with trace τ as before. Then there exists a complex separable Hilbert space $\mathscr{I}(M)^{\perp}$ and a representation $\rho_M : \mathfrak{R} \to \mathfrak{B}(\mathscr{I}(M)^{\perp})$ with the following properties. If $\pi : \mathfrak{R} \to \mathfrak{B}(\mathscr{H})$ is the standard representation constructed above then $0 \subseteqq \mathscr{I}(M)^{\perp} \subsetneqq \mathscr{H}$ and $\rho_M = \pi|_{\mathscr{I}(M)^{\perp}}$ holds.*

Moreover the unitary equivalence class of ρ_M is invariant under orientation-preserving diffeomorphisms of M. Thus the Murray–von Neumann coupling constant[2] of ρ_M is invariant under orientation-preserving diffeomorphisms. Writing $Q_M : \mathscr{H} \to \mathscr{I}(M)^{\perp}$ for the orthogonal projection and taking into account the characterization of $\mathscr{I}(M)^{\perp}$ the coupling constant is equal to $\tau(Q_M) \in [0,1)$ hence $q(M) := \tau(Q_M)$ is a smooth 4-manifold invariant such that $q(M) \in [0,1)$ holds.

Proof. First let us exhibit a representation of $\mathfrak{R}$; we proceed from the Gelfand–Naimark–Segal construction however relative to the already existing standard representation. As we have seen every connected closed oriented smooth 4-manifold M admits an embedding into its von Neumann algebra $\mathfrak{R}$ via (7) and this map looks like $x \mapsto P_x$ *i.e.*, to every point $x \in M$ the map $i_M : M \to \mathfrak{R}$ assigns a projection $P_x \in \mathfrak{R}$ which always satisfies that $0 \neq P_x \neq 1$. Take any Riemannian metric g on M having volume form μ_g over M. Then consider, as a natural choice, the map $F_{M,g} : \mathfrak{R} \to \mathbb{C}$ given by

$$F_{M,g}(A) := \int\limits_{x \in M} i_M^* \tau\big(AP_x\big)\mu_g(x)$$

which is well-defined due to compactness of M. This map is obviously $\mathbb{C}$-linear, ultraweakly continuous and satisfies $F_{M,g}(A^*) = \overline{F_{M,g}(A)}$. Moreover we compute $F_{M,g}(A^*A) = \int_M i_M^* \big\|\widehat{AP_x}\big\|^2 \mu_g(x) \geqq 0$ consequently $F_{M,g}$ is a non-negative functional on $\mathfrak{R}$ and in particular $F_{M,g}(1^*1) > 0$. Therefore an application of the standard inequality $|F_{M,g}(A^*B)|^2 \leqq F_{M,g}(A^*A)F_{M,g}(B^*B)$ implies that $0 \subseteqq I(M,g) \subseteqq \mathfrak{R}$ consisting of those operators $B \in \mathfrak{R}$ for which $F_{M,g}(B^*B) = 0$ is a multiplicative left-ideal in $\mathfrak{R}$ such that $\mathbb{C}^{\times}1 \not\subset I(M,g)$ thus surely $0 \subseteqq I(M,g) \subsetneqq \mathfrak{R}$. In fact $I(M,g)$ is independent of the metric g involved in its definition: if h is another Riemannian metric on M then there exists a positive function $f : M \to \mathbb{R}$ such that

$$0 \leqq F_{M,h}(A^*A) = \int\limits_{x \in M} i_M^* \tau\big(A^*AP_x\big)\mu_h(x) = \int\limits_{x \in M} i_M^* \tau\big(A^*AP_x\big)f(x)\mu_g(x) \leqq F_{M,g}(A^*A)\|f\|_{L^\infty(M)}$$

hence $I(M,g) \subseteqq I(M,h)$ and likewise we see that $I(M,h) \subseteqq I(M,g)$ *i.e.*, $I(M,h) = I(M,g)$. It readily follows from the structure of $F_{M,g}$ that $A \in I(M,g)$ if and only if $AP_x = 0$ for μ_g-a.e. $x \in M$ hence by continuity of (7) for all $x \in M$. This demonstrates that, being P_x surely not invertible, non-trivial solutions in principle are allowed and belong to the infinitesimal ideals $0 \neq \mathfrak{R}(1 - P_x) \subsetneqq \mathfrak{R}$ for all $x \in M$. These are ultraweakly closed left-ideals which in fact are, via diffeomorphisms of M, unitary equivalent. Therefore, considering the things so far merely as a motivation, for a connected closed oriented smooth 4-manifold M hereby we define

$$I(M) := \bigcap_{x \in M} \mathfrak{R}\,(1 - P_x) \tag{9}$$

[2] Also called the $\mathfrak{R}$-*dimension* of a left $\mathfrak{R}$-module hence denoted $\dim_{\mathfrak{R}}$, cf. [2, Chapter 8].

which is by construction an ultraweakly closed left-ideal satisfying $0 \subseteqq I(M) \subsetneqq \mathfrak{R}$ and is invariant under diffeomorphisms (acting as inner automorphisms of $\mathfrak{R}$).

Recall that the standard representation $\pi : \mathfrak{R} \to \mathfrak{B}(\mathscr{H})$ arises via multiplication from the left in $\mathfrak{R}$ on itself. Since $0 \subseteqq I(M) \subsetneqq \mathfrak{R}$ is a left-ideal π restricts to a representation of $\mathfrak{R}$ on the Hilbert space completion $0 \subseteqq \mathscr{I}(M) \subseteqq \mathscr{H}$ of $I(M)$ given by (9). The scalar product on $\mathfrak{R} \subset \mathscr{H}$ looks like $(\hat{A}, \hat{B}) = \tau(AB^*)$ hence satisfies $(\widehat{AB}, \widehat{C}) = (\hat{B}, \widehat{A^*C})$; consequently the standard representation restricts to the orthogonal complementum $0 \subseteqq \mathscr{I}(M)^\perp \subseteqq \mathscr{H}$ as well. Note that $\mathscr{I}(M)^\perp$ as a complete complex vector is isomorphic to $\mathscr{H}/\mathscr{I}(M)$. Therefore in the spirit of the Gelfand–Naimark–Segal construction we want to take the representation of $\mathfrak{R}$ on $\mathscr{I}(M)^\perp$ hence for a given M we introduce the representation $\rho_M : \mathfrak{R} \to \mathfrak{B}(\mathscr{I}(M)^\perp)$ to be simply the restricted representation $\pi|_{\mathscr{I}(M)^\perp}$. The general theory, namely [2, Proposition 8.2.3] says that if $Q_M : \mathscr{H} \to \mathscr{I}(M)^\perp$ is the orthogonal projection then $Q_M \in \mathfrak{R}'$ because for the left $\mathfrak{R}$-module $0 \subseteqq \mathscr{I}(M) \subseteqq \mathscr{H}$ holds *i.e.*, it lies in the standard one (and not in $\mathscr{H} \otimes \ell^2(\mathbb{N})$ as in general). The Murray–von Neumann coupling constant of ρ_M is therefore equal to $\tau(Q_M) \in [0,1]$. Note that the coupling constant depends only on the unitary equivalence class of ρ_M hence in particular it is preserved by orientation-preserving diffeomorphisms of M (which are unitary on $\mathscr{H}$). We conclude that $q(M) := \tau(Q_M) \in [0,1]$ is a smooth invariant of the 4-manifold M.

Concerning an important restriction on the spectrum of q, let $\{U_i\}_{i=1,\dots,m}$ be a finite open covering of M consisting of small open 4-balls and for every $U_i \subset M$ let $U_i \Subset U_i' \subseteqq M$ be the union of those members of the open covering which overlap with U_i as before. By the general theory of ultraweakly closed left-ideals in a von Neumann algebra [12, Theorem 6.1.12], there exist unique projections $P_{U_i'} \in \mathfrak{R}$ such that $\bigcap\limits_{x \in U_i'} \mathfrak{R}(1 - P_x) = \mathfrak{R}(1 - P_{U_i'})$ for every $i = 1, \dots, m$. Hence introducing $I(U_i') := \mathfrak{R}(1 - P_{U_i'})$ the uncountable infinite decomposition in (9) can be coarse-grained to a finite one $I(M) = \bigcap\limits_{i=1}^{m} I(U_i')$. Recall that the geometric meaning of the abstract projection $P_x \in \mathfrak{R}$ assigned to the point $x \in M$ is that it acts as the identity on $\Omega^2(M, x; \mathbb{C})$ too. Taking an arbitrary operator $A \in \bigcap\limits_{x \in U_i'} \mathfrak{R}(1 - P_x) \cap C(M)$, then $AP_x = 0$ implies $0 = (AP_x)\Omega^2(M, x; \mathbb{C}) = A(P_x\Omega^2(M, x; \mathbb{C})) = A\Omega^2(M, x; \mathbb{C})$ for every $x \in U_i'$ or writing equivalently, $0 = A\Omega^2(M, \overline{U_i'}; \mathbb{C})$. Hence $\Omega^2(M, \overline{U_i'}; \mathbb{C}) \subseteqq \bigcap\limits_A \ker A$ moreover we can suppose that $\bigcap\limits_A \ker A \subseteqq \Omega^2(M, \overline{U_i}; \mathbb{C})$ on the smaller subset $U_i \Subset U_i'$. However also $A \in \mathfrak{R}(1 - P_{U_i'}) \cap C(M)$ that is, $AP_{U_i'} = 0$ which implies that $0 = (AP_{U_i'})\Omega^2(M; \mathbb{C}) = A(P_{U_i'}\Omega^2(M; \mathbb{C}))$, dictating the abstract projection $P_{U_i'} \in \mathfrak{R}$ to act on $\Omega^2(M; \mathbb{C})$ too, such that $P_{U_i'}\Omega^2(M; \mathbb{C}) \subseteqq \bigcap\limits_A \ker A \subseteqq \Omega^2(M, \overline{U_i}; \mathbb{C})$. In this way we recognize by referring to the discussion above (7) that $P_{U_i'} \in \mathfrak{R}$ is nothing but an approximating projection of an elementary one $P_{x_i} \in \mathfrak{R}$ such that $x_i \in U_i$. Consequently, if the open covering of M is fine enough, we can suppose that the corresponding projections are already sufficiently close to each other *i.e.*, $[[P_{x_i} - P_{U_i'}]] < 1$ for every $i = 1, \dots, m$ where $[[\cdot]]$ is the norm on $\mathfrak{R}$. However in this situation the projections $P_{x_i}, P_{U_i'} \in \mathfrak{R}$ are unitary equivalent (in general within $\mathfrak{B}(\mathscr{H})$ only), hence the same holds for their corresponding ideals $\mathfrak{R}(1 - P_{x_i}) \subsetneqq \mathfrak{R}$ and $\mathfrak{R}(1 - P_{U_i'}) = I(U_i') \subsetneqq \mathfrak{R}$. We conclude that, in the case of a sufficiently fine but yet finite open covering $\{U_i\}_{i=1}^m$ of M, the already known fact $0 \neq \mathfrak{R}(1 - P_{x_i})$ implies that $0 \neq I(U_i')$ too, for every $x_i \in U_i \Subset U_i'$ and $i = 1, \dots, m$.

Consider $U_1 \subset M$ and take any $0 \neq A_1 \in I(U_1')$. Likewise, take $U_2 \subset M$ and pick $0 \neq A_2 \in I(U_2')$ such that $0 \neq A_1A_2$ (we can assume this by slightly perturbing the factors if necessary). On the one hand, being $I(U_1') \subsetneqq \mathfrak{R}$ a left-ideal, it is acted upon by $\mathfrak{R}$ from the left therefore its Hilbert space completion $\mathscr{I}(U_1') \subseteqq \mathscr{H}$ is a left $\mathfrak{R}$-module. Take again by [2, Proposition 8.2.3] the unique projection $Q_{U_1'} \in \mathfrak{R}'$

satisfying $\mathscr{I}(U_1')=(1-Q_{U_1'})\mathscr{H}$. Then there exists an $\xi\in\mathscr{H}$ such that $\hat{A}_1=(1-Q_{U_1'})\xi$. But this means that there exists a sequence of elements in $\mathfrak{R}$ such that its corresponding sequence $\{\hat{B}_j\}_{j=1,2,\dots}$ converges to ξ in the norm $\|\cdot\|$ on $\mathscr{H}$ as $j\to+\infty$ hence by the continuity of multiplication we get $(1-Q_{U_1'})\hat{B}_j\to\hat{A}_1$. Hence $(1-Q_{U_1'})\widehat{B_jA_2}\to\widehat{A_1A_2}$ as $j\to+\infty$ yielding $\widehat{A_1A_2}\in\mathscr{I}(U_1')$. On the other hand $A_1A_2\in I(U_2')$ implies $\widehat{A_1A_2}\in\mathscr{I}(U_2')$ too. Thus $0\neq\widehat{A_1A_2}\in\mathscr{I}(U_1')\cap\mathscr{I}(U_2')=\mathscr{I}(U_1'\cup U_2')$. Repeating this procedure we come up with a finite product $0\neq\widehat{A_1A_2\dots A_m}\in\bigcap\limits_{i=1}^{m}\mathscr{I}(U_i')=\mathscr{I}(M)$.

Eventually we find $0\subsetneqq\mathscr{I}(M)\subseteqq\mathscr{H}$ hence for the corresponding projection $0<1-Q_M\leqq 1$ demonstrating that $0\leqq q(M)<1$. Therefore essentially the compactness of M implies that $q(M)\in[0,1)$. $\square$

Elements of Jones' subfactor theory (for a summary cf. *e.g.* [2, Section 9.4] or [5, Chapter V.10] or [12, Chapter 19]) offer a computational tool for q just introduced.

Lemma 3.2. *Let M be a connected closed oriented smooth 4-manifold and $q(M)\in[0,1)$ its smooth invariant as before. Then there exists a subfactor $0\neq\mathfrak{I}(M)\subsetneqq\mathfrak{R}$ satisfying $4\leqq[\mathfrak{R}:\mathfrak{I}(M)]<+\infty$ such that $q(M)=1-\frac{4}{[\mathfrak{R}:\mathfrak{I}(M)]}$.*

Proof. We continue working with the left-ideal (9) distilled out of the smooth structure (and the orientation) of M alone. Put

$$\mathfrak{I}(M):=I(M)^*I(M)\ . \tag{10}$$

We assert that $\mathfrak{I}(M)$ is the largest von Neumann subalgebra within $I(M)$ and a non-trivial subfactor within $\mathfrak{R}$. (Meanwhile observe that $I(M)I(M)^*=\mathfrak{R}$ always.) Indeed, by the general theory [12, Theorem 6.1.12], since $0\neq I(M)\subsetneqq\mathfrak{R}$, there exists a unique projection in $\mathfrak{R}$ satisfying $0\neq P_M\neq 1$ such that $I(M)=\mathfrak{R}(1-P_M)$ holds.[3] For $\mathfrak{R}$ is a factor hence multiplication on it is surjective, the equalities $I(M)^*\cap I(M)=(1-P_M)\mathfrak{R}(1-P_M)=I(M)^*I(M)$ follow at once. However by [12, Corollary 3.4.4] we know that $(1-P_M)\mathfrak{R}(1-P_M)\subset\mathfrak{R}$ is a non-trivial subfactor (abstractly isomorphic to $\mathfrak{R}$), hence the assertion also follows.

Let us compute the index $[\mathfrak{R}:\mathfrak{I}(M)]\in\left\{4\cos^2\left(\frac{\pi}{n}\right)\,\middle|\,n=3,4,\dots\right\}\cup[4,+\infty]$. First of all, observe that both $0\neq P_M\neq 1$ and $0\neq 1-P_M\neq 1$ belong to $\mathfrak{R}\cap\mathfrak{I}(M)'$ hence this intersection cannot be trivial; however it is known [12, Corollary 19.2.4] that $\mathfrak{R}\cap\mathfrak{I}(M)'=\mathbb{C}1$ whenever $1\leqq[\mathfrak{R}:\mathfrak{I}(M)]<4$ consequently the index spectrum of (10) is restricted to $[\mathfrak{R}:\mathfrak{I}(M)]\in[4,+\infty]$. Proceeding further, via Footnote 2 we can write that $q(M)=\tau(Q_M)=\dim_{\mathfrak{R}}\mathscr{I}(M)^\perp$ and the standard representation satisfies $\dim_{\mathfrak{R}}\mathscr{H}=1$ hence $\dim_{\mathfrak{R}}\mathscr{I}(M)=1-q(M)$. Moreover, restricting the action of $\mathfrak{R}$ on $\mathscr{I}(M)$ to its subalgebra $\mathfrak{I}(M)$ we can render $\mathscr{I}(M)$ a left $\mathfrak{I}(M)$-module too. Concerning its dimension, recall the general fact [2, Appendix A.2] that in a von Neumann algebra any element can be written as a $\mathbb{C}$-linear combination of at most 4 positive elements *i.e.*, elements of the form B^*B with $B\in\mathfrak{R}$. Taking then $A\in I(M)$ and $P_x\in\mathfrak{R}$ from (7) and inserting $A=b_1B_1^*B_1+\dots+b_4B_4^*B_4$ into $AP_x=0$ and leveraging the linear independence of the terms, we get $B_i^*B_iP_x=0$ hence $0=\left(\widehat{B_i^*B_i},\hat{P}_x\right)=\|\widehat{B_iP_x}\|^2$ yielding $B_iP_x=0$ for all $x\in M$ that is, $B_i\in I(M)$ consequently $B_i^*B_i\in I(M)^*I(M)=\mathfrak{I}(M)$ for all $i=1,2,3,4$. Considering the standard representation of $\mathfrak{I}(M)$ on its own Hilbert space completion having $\mathfrak{I}(M)$-dimension 1, we conclude that $\mathscr{I}(M)$ as a left $\mathfrak{I}(M)$-module decomposes into 4 copies of the standard representation yielding $\dim_{\mathfrak{I}(M)}\mathscr{I}(M)=4$. Thus by [12, Proposition 19.2.3] we compute that $[\mathfrak{R}:\mathfrak{I}(M)]=\frac{\dim_{\mathfrak{I}(M)}\mathscr{I}(M)}{\dim_{\mathfrak{R}}\mathscr{I}(M)}=\frac{4}{1-q(M)}$. Finally, since $q(M)\neq 1$ via Lemma 3.1, we are sure that $[\mathfrak{R}:\mathfrak{I}(M)]\neq+\infty$ hence the result. $\square$

[3] We are using a convention compatible with (9). However note that there is a slight abuse of notation here because we have also introduced M' and a corresponding projection $P_{M'}$ before. Although we set $M'=M$ above note that $P_{M'}\neq P_M$ here because we put $P_{M'}=0$ before but surely $P_M\neq 0$ here. Fortunately this will not cause any confusion in what follows.

Next we collect some basic useful properties of the invariant.

Lemma 3.3. *(Reversing orientation.) If M is a connected closed oriented smooth 4-manifold and $\overline{M}$ is its orientation-reversed form then $q(\overline{M}) = q(M)$.*

(Gluing principle.) Let M and N be two connected closed oriented smooth 4-manifolds and let $M\#N$ be their connected sum. With induced orientation $M\#N$ is a connected closed oriented smooth 4-manifold. Then

$$q(M\#N) = q(M) + q(N) - q(M)q(N)$$

and in particular if $\mathbb{S}^n \subset \mathbb{R}^{n+1}$ is the standard n-sphere then $q(\mathbb{S}^4) = 0$.

(Blow-up.) If M' is a smooth blow-up of M then

$$q(M) \leqq q(M') = q(M) + \big(1 - q(M)\big)t$$

where $0 \leqq t < 1$ is given by $t := q(\mathbb{C}P^2)$.

Proof. The first assertion is obvious from q's construction carried out in the proof of Lemma 3.1.

Concerning the second assertion, let Man^4 denote the category of orientation-preserving diffeomorphism classes of all connected closed oriented smooth 4-manifolds. As a general observation, the q-invariant is a well-defined map from the set of objects of Man^4 into $[0,1) \subset \mathbb{R}$. But these comprise a commutative semigroup with (one possible) unit $\mathbb{S}^4$ under the connected sum operation #. That is, if $X, Y, Z, \mathbb{S}^4 \in \mathsf{Man}^4$ then $X\#Y \cong Y\#X$ and $(X\#Y)\#Z \cong X\#(Y\#Z)$ and $X\#\mathbb{S}^4 \cong X$. Pick $M, N, M\#N \in \mathsf{Man}^4$ and consider the corresponding $q(M), q(N), q(M\#N) \in [0,1)$. Introduce $\bullet : [0,1) \times [0,1) \to [0,1)$ by setting $q(M) \bullet q(N) := q(M\#N)$. The $\bullet$-operation is therefore well-defined having the properties $q(X) \bullet q(Y) = q(Y) \bullet q(X)$ and $(q(X) \bullet q(Y)) \bullet q(Z) = q(X) \bullet (q(Y) \bullet q(Z))$ moreover satisfying $q(X) \bullet q(\mathbb{S}^4) = q(X)$ *i.e.*, $q(\mathbb{S}^4)$ is a unit. These ensure us that $([0,1), \bullet)$ is a unital commutative semigroup and $q : (\mathsf{Man}^4, \#) \to ([0,1), \bullet)$ is a unital semigroup homomorphism. As a specific observation note that q has been defined in Lemma 3.1 as the (continuous) dimension of a closed subspace within the $\mathfrak{R}$-module Hilbert space $\mathscr{H}$; more precisely $q(M) = \tau(Q_M)$ arises through a chain of assignments $M \mapsto \mathscr{I}(M)^\perp \mapsto \dim_{\mathfrak{R}} \mathscr{I}(M)^\perp$. Lemma 3.1 and in particular (9) imply that $\mathscr{I}(M\#N) = \mathscr{I}(M) \cap \mathscr{I}(N)$ for the Hilbert space completion. Introducing the Abelian group $(L(\mathscr{H}), +)$ of closed subspaces with respect to taking sum and hence $0 \in \mathscr{H}$ playing the role of the unit we see that the assignment $M \mapsto \mathscr{I}(M)^\perp$ has the property $\mathscr{I}(M\#N)^\perp = (\mathscr{I}(M) \cap \mathscr{I}(N))^\perp = \mathscr{I}(M)^\perp + \mathscr{I}(N)^\perp$ hence induces a unital semigroup homomorphism $L : (\mathsf{Man}^4, \#) \to (L(\mathscr{H}) \setminus \{\mathscr{H}\}, +)$. Thus q factorizes as

$$q : (\mathsf{Man}^4, \#) \xrightarrow{\ L\ } (L(\mathscr{H}) \setminus \{\mathscr{H}\}, +) \xrightarrow{\ \dim_{\mathfrak{R}}\ } ([0,1), \bullet)$$

consequently the unique plain dimension function on subspaces as it is, must in fact be an Abelian unital semigroup homomorphism too hence its properties impose constraints on its target structure. Recall the q-invariant when evaluated on a connected sum has been written as $q(M\#N) = q(M) \bullet q(N)$ expressing that it depends only on $q(M)$ and $q(N)$. Putting $\mathscr{V} = \mathscr{I}(M)^\perp$ and $\mathscr{W} = \mathscr{I}(N)^\perp$ and knowing that the dimension function behaves like $\dim_{\mathbb{R}}(\mathscr{V} + \mathscr{W}) = \dim_{\mathfrak{R}} \mathscr{V} + \dim_{\mathfrak{R}} \mathscr{W} - \dim_{\mathfrak{R}}(\mathscr{V} \cap \mathscr{W})$ forces to write $q(M) \bullet q(N) = q(M) + q(N) - f\big(q(M), q(N)\big)$ and the unknown function is strongly determined by the geometric properties of intersection of subspaces. Namely we know that $f : [0,1) \times [0,1) \to [0,1)$ such that $f(s,t) = f(t,s)$ and $s \mapsto f(s,t)$ is linear, $f(0,t) = 0$ and $\lim_{s \to 1} f(s,t) = t$. The function $f(s,t) = st$ solves these constraints. This forces the Abelian semigroup multiplication law on $[0,1)$ to look like $s \bullet t = s + t - st$ with $0 \in [0,1)$ being the unit hence yielding the shape for $q(M\#N)$ moreover constraining $q(\mathbb{S}^4) = 0$ as stated.

Finally, since the blow-up in the smooth category $(\mathsf{Man}^4, \#)$ is by definition given by $M' := M\#\overline{\mathbb{C}P^2}$ the result follows if we put $t := q(\overline{\mathbb{C}P^2}) = q(\mathbb{C}P^2)$ in the connected sum formula. $\square$

Proof of Theorem 1.2. Putting together Lemmata 3.1, 3.2 and 3.3 the theorem follows. □

An excursus on the invariant q. Let us take a closer look on the introduced smooth 4-manifold invariant concerning its effective computability. The general experience is that the more sensitive an invariant is, the less computable it is. By techniques borrowed from 4-manifold theory and the gluing principle above, the following non-trivial but quite insensitive behaviour of q shows up in the finite fundamental group situation.

Lemma 3.4. *Let M' and M'' be closed connected orientable smooth 4-manifolds having finite fundamental groups. If M' and M'' are homeomorphic then $q(M')=q(M'')$.*

In fact if M is as above then

$$q(M)=1-|\pi_1(M)|(1-t)^{\mathrm{rk}(\pi_2(M))}$$

with $t=q(\mathbb{C}P^2)$ as before and the condition $0\leqq q(M)<1$ makes sure that $0<t<1$.

Proof. Assume first that both M' and M'' are closed, connected, simply connected (hence orientable) and are homeomorphic. Then by *e.g.* [10, Theorem 9.1.12] there exists an integer $k\geqq 0$ such that $M'\#k(\mathbb{C}P^1\times\mathbb{C}P^1)\cong M''\#k(\mathbb{C}P^1\times\mathbb{C}P^1)$ yielding $q(M'\#k(\mathbb{C}P^1\times\mathbb{C}P^1))=q(M''\#k(\mathbb{C}P^1\times\mathbb{C}P^1))$. Introducing $s:=q(k(\mathbb{C}P^1\times\mathbb{C}P^1))$ and applying the gluing principle from Lemma 3.3 we obtain an equality $q(M')+s-q(M')s=q(M'')+s-q(M'')s$. Finally leveraging the invertability of the map $r\mapsto r+s-rs$ when $s<1$, we come up with $q(M')=q(M'')$.

Concerning the explicit formula in the simply connected realm first, with some $s\in[0,1)$ take the recursive sequence

$$R_0(s):=0\,,\,R_1(s):=s\,,\,\dots\,,\,R_k(s):=s+R_{k-1}(s)-sR_{k-1}(s)\,,\,\dots$$

for all $k=0,1,2,\dots$ representing the semigroup $\{0\}\cup\mathbb{N}$ inside $[0,1)$ and put $t:=q(\mathbb{C}P^2)=q(\overline{\mathbb{C}P^2})$. Now if M_1 and M_2 are connected, closed, simply connected, smooth then there exist integers $k_1,l_1\geqq 0$ and $k_2,l_2\geqq 0$ such that $M_1\#k_1\mathbb{C}P^2\#l_1\overline{\mathbb{C}P^2}\cong M_2\#k_2\mathbb{C}P^2\#l_2\overline{\mathbb{C}P^2}$ (cf. *e.g.* [10, Theorem 9.1.14]) which gives again that $q(M_1\#k_1\mathbb{C}P^2\#l_1\overline{\mathbb{C}P^2})=q(M_2\#k_2\mathbb{C}P^2\#l_2\overline{\mathbb{C}P^2})$. Then by the gluing principle

$$q(M_1)+R_{k_1+l_1}(t)-q(M_1)R_{k_1+l_1}(t)=q(M_2)+R_{k_2+l_2}(t)-q(M_2)R_{k_2+l_2}(t)\;.$$

Let $M_1:=M$ be arbitrary and $M_2:=\mathbb{S}^4$ hence $q(M_2)=0$. Then we can suppose that $k_1+l_1\leqq k_2+l_2$ therefore $q(M)+R_{k_1+l_1}(t)-q(M)R_{k_1+l_1}(t)=R_{k_2+l_2}(t)$ from which again, since $t<1$, by invertability we find $q(M)=R_{k_2+l_2-k_1-l_1}(t)$ hence setting $n:=k_2+l_2-k_1-l_1\geqq 0$ we get $q(M)=R_n(t)$. It is clear from the proof that here $n=b_2(M)$ i.e., $n=\mathrm{rk}(\pi_2(M))$ by Hurewicz's theorem, moreover it is easy to see that $R_k(s)=1-(1-s)^k$ is the solution of the above recursion. Hence inserting $k:=\mathrm{rk}(\pi_2(M))$ and $s:=t$ and knowing that $|\pi_1(M)|=1$ we end up with the stated formula for $q(M)$ in the simply connected situation.

Concerning the non-simply connected case, first recall the well-known fact that every finite group G appears as the fundamental group of some closed, connected and orientable smooth 4-manifold M i.e., $\pi_1(M)\cong G$. Consider the universal covering $\widetilde{M}$ which is therefore remains closed, and is connected and simply connected. Then $\widetilde{M}/G\cong M$ consequently $\widetilde{M}$ is acted upon freely by G and M can be identified with an open subset of the universal covering space, namely one particular fundamental domain $F(M)\subset\widetilde{M}$ of this G-action on $\widetilde{M}$. First: this inclusion, by referring to (9), implies $I(\widetilde{M})\subset I(F(M))$ hence via (10) we also obtain an inclusion $\mathfrak{I}(\widetilde{M})\subset\mathfrak{I}(F(M))$ of von Neumann algebras. Second: on the one hand as G permutes the fundamental domains, it acts via outer $*$-automorphisms on $\mathfrak{I}(F(M))$ exhibiting a fixed-point subalgebra $\mathfrak{I}(F(M))^G \subset \mathfrak{I}(F(M))$; on the other hand the obvious inclusion $G \subset \mathrm{Diff}^+(\widetilde{M})$, compared with (9) which says that $I(\widetilde{M})$ is an intersection of pointwise ideals permuted by $\mathrm{Diff}^+(\widetilde{M})$ hence $I(\widetilde{M})$ is pointwisely fixed by $\mathrm{Diff}^+(\widetilde{M})$ thus by (10) so is $\mathfrak{I}(\widetilde{M})$, gives $\mathfrak{I}(\widetilde{M}) = \mathfrak{I}(F(M))^G$. Third: the existence of an orientation-preserving diffeomorphism between $F(M)$ and an open subset of M having measure-zero complementum hence an isomorphism $I(F(M)) \cong I(M)$, implies an isomorphism $\mathfrak{I}(F(M)) \cong \mathfrak{I}(M)$ of von Neumann algebras and we already know that the latter is a subfactor of $\mathfrak{R}$ hence so is $\mathfrak{I}(F(M))$. Now putting together these three pieces of data and referring to [12, Corollary 19.1.6] we observe that $\mathfrak{I}(\widetilde{M}) \subset \mathfrak{I}(F(M))$ is an inclusion of subfactors having index $[\mathfrak{I}(F(M)) : \mathfrak{I}(\widetilde{M})] = |G|$. Consequently, by referring to Lemma 3.2, we compute $q(\widetilde{M}) = 1 - \frac{4}{[\mathfrak{R}:\mathfrak{I}(\widetilde{M})]} = 1 - \frac{4}{[\mathfrak{R}:\mathfrak{I}(F(M))][\mathfrak{I}(F(M)):\mathfrak{I}(\widetilde{M})]} = 1 - \frac{4}{[\mathfrak{R}:\mathfrak{I}(M)]|G|}$ and then multiply both sides with $1 < |G| < +\infty$ and insert $q(M) = 1 - \frac{4}{[\mathfrak{R}:\mathfrak{I}(M)]}$ too. This way we find that $q(M) = 1 + |G|(q(\widetilde{M}) - 1)$. Finally plugging $G \cong \pi_1(M)$ as well as the already known expression $q(\widetilde{M}) = 1 - (1-t)^{\mathrm{rk}(\pi_2(\widetilde{M}))}$ from the simply connected case and recalling that covering implies an isomorphism $\pi_2(\widetilde{M}) \cong \pi_2(M)$ we come up with the desired formula in the non-simply connected case too. □

As an important task next we compute the number $0 < t < 1$ apprearing in both Lemmata 3.3 and 3.4 and is assigned to $\mathbb{C}P^2$.

Lemma 3.5. *An equality* $q(\mathbb{C}P^2) = \frac{5}{9}$ *holds hence* $t = \frac{5}{9}$. *Thus* $q(M) = 1 - |\pi_1(M)|\left(\frac{2}{3}\right)^{2\,\mathrm{rk}(\pi_2(M))}$ *in Lemma 3.4. Moreover the condition* $0 \leqq q(M)$ *gives rise to an inequality* $|\pi_1(M)| \leqq \left(\frac{3}{2}\right)^{2\,\mathrm{rk}(\pi_2(M))}$ *whenever M is a closed, connected, orientable smooth* 4*-manifold having finite fundamental group.*

Proof. We have seen that the action $x \mapsto \Phi(x)$ of a diffeomorphism on a point $x \in M$ induces the unitary action $P_x \mapsto (\Phi^{-1})^* P_x \Phi^*$ on the corresponding projection $P_x \in \mathfrak{R}$ by (7) hence this action is an inner $*$-automorphism of $\mathfrak{R}$. In this way we obtain a commutative diagram

$$\begin{array}{ccc} \mathfrak{R} & \xrightarrow{\mathrm{Ad}_{(\Phi^{-1})^*}} & \mathfrak{R} \\ {\scriptstyle i_M}\uparrow & & \uparrow{\scriptstyle i_M} \\ M & \xrightarrow{\Phi} & M \end{array}$$

which governs the inclusion of $\mathrm{Diff}^+(M)$ as a subgroup of inner $*$-automorphisms into $\mathrm{Aut}\,\mathfrak{R}$. It is clear that the corresponding subfactor $\mathfrak{I}(M)$ from Lemma 3.2 must be invariant within $\mathfrak{R}$ under diffeomorphisms *i.e.*, is preserved by $\mathrm{Diff}^+(M)$ acting as inner $*$-automorphisms on $\mathfrak{R}$. Assume furthermore that there exists a *compact* subgroup $G \subsetneqq \mathrm{Diff}^+(M)$ which acts transitively on M. Thus taking into account the decomposition in (9) we can see by picking any point $x_0 \in M$ that

$$I(M) = \bigcap_{x \in M} \mathfrak{R}(1 - P_x) = \bigcap_{\Phi \in \mathrm{Diff}^+(M)} \mathfrak{R}\big(1 - P_{\Phi(x_0)}\big) = \bigcap_{\Psi \in G} \mathfrak{R}\big(1 - P_{\Psi(x_0)}\big)$$

that is, the ideal already arises as taking intersection under the subgroup only. Moreover being an intersection, we have already observed that $I(M)$ in fact is pointwisely fixed by G hence by (10) so is $\mathfrak{I}(M)$. Therefore an effective form of the invariance condition is that $\mathfrak{I}(M)$ belongs to the *fixed point subalgebra* $\mathfrak{R}^G$ of a *compact* group G operating as inner $*$-automorphisms on $\mathfrak{R}$, dictated by the diagram above.

Consider now the Fubini–Study metric g on the complex projective plane $\mathbb{C}P^2$. It is a classical fact that this is a homogeneous Riemannian 4-manifold satisfying the Einstein condition with a non-zero cosmological constant. The isometry group $\mathrm{Iso}^+(\mathbb{C}P^2,g)\subsetneqq\mathrm{Diff}^+(\mathbb{C}P^2)$ of the Fubini–Study metric is large enough to act transitively along $\mathbb{C}P^2$. A way to implement $\mathrm{Iso}^+(\mathbb{C}P^2,g)\cong\mathrm{PU}(3)$ into $\mathfrak{B}(\mathscr{H})$ to meet our demands is to take conjugations by 3×3 unitary matrices on $\mathfrak{R}\cong\mathfrak{M}_3(\mathfrak{R})\cong\mathfrak{M}_3(\mathfrak{I}(\mathbb{C}P^2))$. Consequently $[\mathfrak{R}:\mathfrak{I}(\mathbb{C}P^2)]=[\mathfrak{M}_3(\mathfrak{I}(\mathbb{C}P^2)):\mathfrak{I}(\mathbb{C}P^2)]=3^2=9$ demonstrating by Lemma 3.2 that $q(\mathbb{C}P^2)=1-\frac{4}{[\mathfrak{R}:\mathfrak{I}(\mathbb{C}P^2)]}=1-\frac{4}{9}=\frac{5}{9}=t$ as stated. □

Lemmata 3.4 and 3.5 unfortunately make sure that in the finite-fundamental-group regime at least, q is not really sensitive because it depends only on two simple topological data. In fact q gets less-and-less injective as the sizes of the 1st and the 2nd homotopy groups of a topological 4-manifold carrying smooth structure(s) increases. On the contrary, as these parameters approach zero, q has a sharp content: an application to $\mathbb{S}^4$ satisfying $q(\mathbb{S}^4)=0$ implies that if the smooth four dimensional Poincaré conjecture fails then q is not injective at zero too. A stronger assertion is the following.

Lemma 3.6. *Consider q as a restricted map from (the objects of) the category Man^4_{fin} of isomorphism classes of all connected, closed, orientable, smooth 4-manifolds having finite fundamental group, into $[0,1)\subset\mathbb{R}$. Then q is injective at zero if and only if the smooth Poincaré conjecture holds i.e., the 4-sphere admits a unique smooth structure.*

Proof. If S^4 is a differentiable manifold homeomorphic to $\mathbb{S}^4$ then $S^4\in\mathsf{Man}^4_{fin}$ such that $\pi_1(S^4)=1$ and $\pi_2(S^4)=0$ therefore Lemma 3.4 implies $q(S^4)=1-1\cdot(1-t)^0=0$. Assume that $q|_{\mathsf{Man}^4_{fin}}$ is injective at zero. Then $S^4\cong\mathbb{S}^4$ and the 4 dimensional smooth Poincaré conjecture follows.

Conversely, take $S^4\in\mathsf{Man}^4_{fin}$ such that $q(S^4)=0$. Then by Lemmata 3.4 and 3.5 we conclude that $q(S^4)=1-|\pi_1(S^4)|\left(\frac{4}{9}\right)^{\mathrm{rk}(\pi_2(S^4))}=0$ hence necessarily $|\pi_1(S^4)|=1$ *i.e.*, $\pi_1(S^4)=1$ and $\mathrm{rk}(\pi_2(S^4))=0$. Hurewicz's theorem and the universal coefficient formula together imply that $H_2(S^4;\mathbb{Z})=0$ *i.e.*, S^4 has trivial intersection form. The validity of the 4 dimensional topological Poincaré conjecture [9] thus makes sure that S^4 is homeomorphic to $\mathbb{S}^4$. Assume that the smooth 4 dimensional Poincaré conjecture is true. Then $S^4\cong\mathbb{S}^4$ hence $q|_{\mathsf{Man}^4_{fin}}$ is injective at zero. □

The content of Lemma 3.6 *i.e.*, the interpretation of the preimage $q^{-1}(0)\subset\mathsf{Man}^4_{fin}$ can be understood geometrically as the description of the "fiber" of q considered as a map from Man^4_{fin} into $[0,1)\subset\mathbb{R}$. This strongly motivates to move from understanding q in individual cases $M\in\mathsf{Man}^4_{fin}$ to its global behaviour over the whole (objects of) Man^4_{fin}. Our next observation in this direction is the following:

Lemma 3.7. *Assume that the image of q as a restricted map from (the objects of) Man^4_{fin} as above into the semi-open real interval $[0,1)$ is everywhere dense. Then q is nowhere injective and in fact the cardinalities of its preimage sets are always infinite.*

Proof. For $r\in[0,1)$ let $|q^{-1}(r)|=0,1,2,\dots,+\infty$ denote the cardinality of the preimage set of q at the point r *i.e.*, the collection of all pairwise non-diffeomorphic 4-manifolds having q-invariants equal to r. Here $|q^{-1}(r)|=0$ means $q^{-1}(r)=\emptyset$ *i.e.*, r is not in the range of q; while the other extreme case $|q^{-1}(r)|=+\infty$ means there exist infinitely many pairwise non-diffeomorphic smooth 4-manifolds, perhaps homeomorphic or not, having q-invariants all equal to r. Consider the subset $S:=\{s\in[0,1)\,|\,|q^{-1}(s)|>1\}\subseteqq\mathrm{im}\,q\subseteqq[0,1)$. We assert that $S=\mathrm{im}\,q$. The proof is based on a modified open-closed argument adapted to the possibility that $\mathrm{im}\,q$ is totally disconnected in $[0,1)$.

We observe first that S is not empty. To be explicit let M be a fixed $K3$ surface; it is known for a long time [10, Corollary 3.3.7] that the simply connected topological 4-manifold underlying $K3$

carries countably infinitely many different smooth structures. Consequently referring to Lemma 3.4 and putting $s := 1-\left(\frac{4}{9}\right)^{b_2(K3)} = 1-\left(\frac{4}{9}\right)^{22}$ we know that $|q^{-1}(s)| = +\infty$ yielding $s \in S$.

Next we prove that S is open in the following sense: for every inner point $s \in (0,1)\cap S$ there exists $\varepsilon > 0$ such that $t \in S$ for every $t \in (s-\varepsilon, s+\varepsilon)\cap \operatorname{im} q$ moreover $s \pm \varepsilon \in \operatorname{im} q$. First: consider the projection $P_M \in \mathfrak{R}$ having been defined by $I(M) = \mathfrak{R}(1-P_M)$ in the proof of Lemma 3.2. There is a straightforward equality $(1-P_M)\mathfrak{R}(1-P_M) = (1-P_M)\mathfrak{I}(M)(1-P_M)$ thus by [12, Proposition 19.2.3] again, we compute $1 = [(1-P_M)\mathfrak{R}(1-P_M) : (1-P_M)\mathfrak{I}(M)(1-P_M)] = \tau(1-P_M)T'(1-P_M)[\mathfrak{R} : \mathfrak{I}(M)]$ where T' is the normalized trace on $\mathfrak{I}(M)'$. But $T'(1-P_M) = T(J(1-P_M)J) = T(1-P_M) = 1$ since $1-P_M$ obviously acts as the unit on $\mathfrak{I}(M) = (1-P_M)\mathfrak{R}(1-P_M)$. Inserting this and the formula for $q(M)$ in Lemma 3.2 we get $q(M) = 4\tau(P_M)-3$. Second: put the ultraweak topology on $\mathfrak{R}$. On the one hand the trace $\tau : \mathfrak{R} \to \mathbb{C}$ is then continuous. On the other hand let us topologize (the set of objects in) Man^4_{fin} by pulling back this topology from the dense subspace of projections of $\mathfrak{R}$ by the aid of the map $M \mapsto P_M$. This map is therefore continuous by construction. Hence $M \mapsto P_M \mapsto 4\tau(P_M)-3 = q(M)$ is also a continuous function from Man^4_{fin} into $[0,1)$. Therefore the assignment $M \mapsto q(M)$ looks like a branched covering over $\operatorname{im} q$ consequently $t \mapsto |q^{-1}(t)|$ is lower semicontinuous: for every inner point $s \in (0,1)\cap \operatorname{im} q$ there exists $\varepsilon > 0$ such that $|q^{-1}(t)| \geqq |q^{-1}(s)|$ whenever $t \in (s-\varepsilon, s+\varepsilon)\cap \operatorname{im} q$. Now applying this to any $s \in (0,1)\cap S$ we get $t \in S$ for every $t \in (s-\varepsilon, s+\varepsilon)\cap \operatorname{im} q$ and notice that such t's always exist since by assumption $\operatorname{im} q$ is everywhere dense within $[0,1)$. Moreover passing to a smaller $\varepsilon > 0$ if necessary, we can suppose in addition that the endpoints satisfy $s \pm \varepsilon \in \operatorname{im} q$. Hence S is open within $\operatorname{im} q$ and note that the complementum of this type of open interval is closed within $\operatorname{im} q$.

Finally we prove closedness in the usual sense: for every sequence $\{s_i\}_{i=1,2,\dots}$ within S the convergence to a point $t \in \operatorname{im} q$ implies $t \in S$. First: take a fixed object $M \in \mathsf{Man}^4_{fin}$. It follows from the strong*-limit-decomposition (8) of an automorphism of $\mathfrak{R}$ with respect to M, together with the strong*-closedness of the embedding (7) of M into $\mathfrak{R}$, that M can be identified with the orbit of any of its elementary projection $P_x \in \mathfrak{R}$ under $\operatorname{Aut}\mathfrak{R}$. Consequently if $M, N \in \mathsf{Man}^4_{fin}$ are not diffeomorphic then their corresponding embeddings (7) are disjoint within $\mathfrak{R}$. In particular if $x \in M$ and $y \in N$ then their $P_x, P_y \in \mathfrak{R}$ via (7) are not unitary equivalent hence so are their ideals $I(M), I(N) \subset \mathfrak{R}$ given by taking intersections as in (9); thus their corresponding projections $1-P_M, 1-P_N \in \mathfrak{R}$ as above, are also not unitary equivalent. Therefore $[[(1-P_M)-(1-P_N)]] = [[P_M - P_N]] = 1$ where $[[\cdot]]$ is the norm on $\mathfrak{R}$. Consequently at least one of the pairs $(1-P_M, 1-P_N)$ or (P_M, P_N) contains non-trivial orthogonal projections. Since the sum of the traces of orthogonal projections cannot exceed $1 \in [0,1]$ then from $0 \leqq q(M) = 4\tau(P_M)-3 < 1$ *i.e.*, $\frac{3}{4} \leqq \tau(P_M) < 1$ and likewise for N, we infer that $1-P_M, 1-P_N$ are orthogonal yielding $(1-P_M)(1-P_N) = 0 = (1-P_N)(1-P_M)$. Second: let $\{s_i\}_{i=1,2,\dots}$ be a sequence in S converging to a point $t \in \operatorname{im} q$. Then for every $i = 1,2,\dots$ we know that $|q^{-1}(s_i)| > 1$ therefore there exist $M_i, N_i \in \mathsf{Man}^4_{fin}$ satisfying $q(M_i) = s_i = q(N_i)$ such that $M_i \not\cong N_i$ in Man^4_{fin} hence $(1-P_{M_i})(1-P_{N_i}) = 0 = (1-P_{N_i})(1-P_{M_i})$ in $\mathfrak{R}$. Taking the limit $i \to +\infty$ by continuity of q there exist $M, N \in \mathsf{Man}^4_{fin}$ such that $q(M) = t = q(N)$. However the corresponding projections are non-trivial and by the constructed ultraweak continuity of the assignment $M \mapsto P_M$ (in which multiplication is continuous) they yet satisfy $(1-P_M)(1-P_N) = 0 = (1-P_N)(1-P_M)$. Therefore $P_M \neq P_N$ yielding $M \not\cong N$. Consequently $|q^{-1}(t)| > 1$ continues to hold *i.e.*, $t \in S$. Thus S is closed within $\operatorname{im} q$ as well.

Thus S is not empty, S is open in $\operatorname{im} q$ with closed complementum morever S is closed in $\operatorname{im} q$ as well hence our conclusion is therefore that $S = \operatorname{im} q$ and the assertion follows. We also conclude that the cardinality of $q^{-1}(t)$ for every $t \in \operatorname{im} q \subsetneqq [0,1)$ is in fact infinite. □

An important question is therefore whether or not the image of Man^4_{fin} under q is dense within the real semi-open interval $[0,1)$. At this point we can make only a quite straightforward observation as follows.

Lemma 3.8. *Consider* Man^4_{fin} *as in Lemma 3.7 and suppose that its set of objects possesses at least one of the two properties below:*

(i) There is a sequence $\{M_m\}_{m=1,2,\dots} \subset \mathsf{Man}^4_{fin}$ *of 4-manifolds such that the associated sequence of their homotopy groups satisfies* $|\pi_1(M_m)| \to +\infty$ *and* $\left|\frac{\log|\pi_1(M_m)|}{\log\frac{9}{4}} - \mathrm{rk}(\pi_2(M_m))\right| \to 0$ *as* $m \to +\infty$*;*

(ii) Take an inner point $r \in (0,1)$ *and an associated sequence* $\{(m_n, n)\}_{n=1,2,\dots}$ *of pairs of integers with* $m_n > 0$ *satisfying* $(m_n - 1)(\frac{2}{3})^{2n} \leqq 1 - r \leqq m_n(\frac{2}{3})^{2n}$ *or* $m_n(\frac{2}{3})^{2n} \leqq 1 - r \leqq (m_n+1)(\frac{2}{3})^{2n}$*. Then for every real number* $r \in (0,1)$ *and pair of positive integers* (m_n, n) *from its approximating sequence there exists an* $M \in \mathsf{Man}^4_{fin}$ *such that* $\big(|\pi_1(M)|, \mathrm{rk}(\pi_2(M))\big) = (m_n, n)$*.*

Then it follows that $\operatorname{im} q$ *is dense in* $[0,1)$*.*

Proof. (i) Since $q(M) = 1 - |\pi_1(M)|\big(\frac{2}{3}\big)^{2\,\mathrm{rk}(\pi_2(M))}$ by Lemma 3.5 and $0 \leqq q(M) < 1$ by Lemma 3.1, taking $i = 1,2,\dots$ and $j = 1,2,\dots$ we examine rational numbers of the form $0 < 1 - i\big(\frac{2}{3}\big)^{2j} < 1$. For every $m > 0$ there exist lot of finitely presented groups G such that $|G| = m$ as well as $M_m \in \mathsf{Man}^4_{fin}$ so that $\pi_1(M_m) \cong G$ hence $|\pi_1(M_m)| = m$. Put $n_m := \mathrm{rk}(\pi_2(M_m))$ thus $\big(|\pi_1(M_m)|, \mathrm{rk}(\pi_2(M_m))\big) = (m, n_m)$. Passing to a subsequence if necessary, *suppose* that there exists $M_1, M_2, \dots \in \mathsf{Man}^4_{fin}$ having the property $q(M_m) = 1 - m\big(\frac{2}{3}\big)^{2n_m} \to 0$ as $m \to +\infty$. Observe that this implies $n_m \to +\infty$ in an approriate way too; more precisely writing $m\big(\frac{2}{3}\big)^{2n_m} = \mathrm{e}^{\log m - n_m \log\frac{9}{4}}$ we see that $n_m \sim \frac{\log m}{\log\frac{9}{4}}$ as $m \to +\infty$. Given a fixed inner point $r \in (0,1)$ by assumption there exists $M_m \in \mathsf{Man}^4_{fin}$ so that $0 < q(M_m) \leqq r$. For a fixed m the existence $M_m \in \mathsf{Man}^4_{fin}$ implies that its smooth blow-up $M'_m = M_m \# \overline{\mathbb{C}P^2}$ exists in Man^4_{fin} too, and $q(M_m) < q(M'_m)$ via Lemmata 3.3 and 3.5. Thus taking blow-ups of M_m in $0 < k < +\infty$ distinct points, we can arrange $q\big(M_m^{(k-1)}\big) \leqq r \leqq q\big(M_m^{(k)}\big)$. To close this scissor about r, observe that in fact $q(M_m^{(\ell)}) = 1 - m\big(\frac{2}{3}\big)^{2(n_m+\ell)}$ via the gluing formula in Lemma 3.3 hence an m-independent estimate $q\big(M_m^{(k)}\big) - q\big(M_m^{(k-1)}\big) = m\big(\frac{4}{9}\big)^{n_m+k-1}\big(1 - \frac{4}{9}\big) \leqq \frac{5}{9}\big(\frac{4}{9}\big)^{k-1}$ holds; thus at the expense of increasing m and then k we can also arrange that $0 < q\big(M_m^{(k)}\big) - q\big(M_m^{(k-1)}\big) < \varepsilon$. Consequently $r \in (0,1)$ can be approximated as pleased demonstrating that $\operatorname{im} q$ is indeed dense in $[0,1)$ hence the result.

(ii) This assertion is obvious (and has been stated only because it is dual to the first assertion in an appropriate sense). $\square$

If the hypothetical sequence of 4-manifolds in Lemma 3.8 exists then $\operatorname{im} q \subsetneqq [0,1)$ is dense consequently $|q^{-1}(0)| = +\infty$ by Lemma 3.7 hence the smooth Poincaré conjecture fails by Lemma 3.6. Likewise $|q^{-1}\big(\frac{5}{9}\big)| = +\infty$ and one can easily check via Lemma 3.5 and Freedman's classification [9] that topologically $q^{-1}\big(\frac{5}{9}\big) = \big\{\mathbb{C}P^2, \overline{\mathbb{C}P^2}\big\}$ over Man^4_{fin} hence at least one of these spaces carries infinitely many smooth structures too.

Proof of Theorem 1.3. Putting together Lemmata 3.4, 3.5, 3.6, 3.7 and 3.8 the theorem follows. $\square$

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