Gentle Statistical Mechanics — Third HW problem set

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▷ **Exercise 1.** Below are three Markov chains described in words, along with three distributions. Write down the state spaces of the chains, their transition probabilities, and prove that the corresponding distributions are stationary.

(a)• There are k balls in n urns arranged in a circle. Every second, we randomly select one of the k balls and move it to the adjacent urn in a clockwise direction, provided that the destination urn is empty. If the destination urn is not empty, we do nothing.

Fermi-Dirac distribution: k balls are randomly distributed among $n \geq k$ urns such that each urn contains at most one ball.

(b)• There are k balls in n urns arranged in a circle. Every second, we randomly select one of the k balls and move it to the adjacent urn in a clockwise direction.

Maxwell-Boltzmann distribution: k distinguishable balls are randomly distributed among n urns.

(c)• There are n urns arranged in a circle. Every second, we randomly select one of the urns. If there is a ball in the selected urn, we move one ball from it to the adjacent urn in a clockwise direction.

Bose-Einstein distribution: k indistinguishable balls are randomly distributed among n urns.

A Markov chain (V, P) is called irreducible if for every $x, y \in V$ there is some n such that $p_n(x, y) = \mathbf{P}[\, X_n = y \mid X_0 = x \,] > 0$. It is called aperiodic if $\mathsf{period}(x) := \gcd\{n : p_n(x, x) > 0\} = 1$ for every $x \in V$. It takes a bit of number theory to show that in any finite irreducible aperiodic Markov chain there is an r such that $p_r(x, y) > 0$ for every x, y simultaneously.

▷ **Exercise 2.** For any finite irreducible aperiodic Markov chain (V, P) with stationary measure π, using the r above, show the following:

(a)• There is a $\delta \in [0, 1)$ such that $P^r = (1 - \delta)\Pi + \delta Q$, where Π is the square matrix with each row equalling π, and Q is some other transition matrix.

(b)• Show by induction that $P^{nr} = (1 - \delta^n)\Pi + \delta^n Q^n$ for every $n \geq 1$.

(c)• Conclude that $\lim_{n\to\infty} p_n(x, y) = \pi(y)$ for every x, y, and for any initial distribution ν, we have $\|\nu P^n - \pi\|_{\mathrm{TV}} \to 0$ for the total variation distance introduced in the previous exercise sheet.

▷ **Exercise 3.•** When the New York Times in 1990 reported on 7 riffle shuffles being enough for a deck to be well-mixed, they wrote: "By saying that the deck is completely mixed after seven shuffles, Dr. Diaconis and Dr. Bayer mean that every arrangement of the 52 cards is equally likely or that any card is as likely to be in one place as in another." True or false: Let μ be a distribution on the symmetric group S_n such that when $\sigma \in S_n$ is chosen according to μ, we have $\mathbf{P}[\, \sigma(i) = j \,] = 1/n$ for every $i, j \in \{1, \dots, n\}$. Then μ is uniform on S_n.

▷ **Exercise 4.•** When the New York Times in 1990 reported on 7 riffle shuffles being enough for a deck to be well-mixed, they wrote: "By saying that the deck is completely mixed after seven shuffles, Dr. Diaconis and Dr. Bayer mean that every arrangement of the 52 cards is equally likely or that any card is as likely to be in one place as in another." True or false: Let μ be a distribution on the symmetric group S_n such that when $\sigma \in S_n$ is chosen according to μ, we have $\mathbf{P}[\, \sigma(i) = j \,] = 1/n$ for every $i, j \in \{1, \dots, n\}$. Then μ is uniform on S_n.

▷ **Exercise 5.** Let $G(V, E)$ be a connected infinite graph.

(a)$^\bullet$ Show that if simple random walk started at some $o \in V$ visits o infinitely often almost surely, then the walk started at any $x \in V$ visits any given $y \in V$ infinitely often, almost surely. Consequently, recurrence is a property solely of the graph.

(b)$^\bullet$ If the graph is transient, then the walk visits any given finite set only finitely many times almost surely.

▷ **Exercise 6.**$^{\bullet\bullet}$ Give an example of an iid random walk on $\mathbb{Z}$ with symmetric jump distribution that is transient. (Hint: the explicit example you will find on the internet is missing the main step in the calculation, so it's no good. But you don't need to give the jump probabilities explicitly. Instead, use that simple random walk on $\mathbb{Z}^2$ is recurrent, on $\mathbb{Z}^3$ is transient, and you don't need any further calculations.)

▷ **Exercise 7.** The Hungarian Media Police has observed five possible TV-watching behaviours that people may have: (1) never watches the TV; (2) watches only state channels; (3) regularly watches the TV; (4) TV-addict; (5) brain-dead. The transitions between these states may be modelled by a Markov chain, with the following transition matrix:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0.6 & 0 & 0.4 & 0 & 0 \\ 0.3 & 0 & 0.3 & 0.1 & 0.3 \\ 0 & 0 & 0.4 & 0.4 & 0.2 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

In particular, nobody *becomes* a state channel fan — one has to be born like that.

(a)$^\bullet$ If one starts as a state channel fan, what is the probability that they end up brain-dead?

(b)$^\bullet$ What is the expected time for a state channel fan to reach a terminal state: to quit TV completely, or to become brain-dead?

Hints: As we briefly discussed, (a) is just solving a system of linear equations that come from conditioning on the first step. Part (b) is very similar, just don't forget that the first step takes time.

▷ **Exercise 8.**$^\bullet$ Consider simple symmetric random walk on $\mathbb{Z}$. Starting from 0, calculate the expected time to reach 1. (Hint: write a recursion.)

In class we talked about stationary and reversible distributions (probability measures) only, but these notions ($\pi P = \pi$ for stationarity, and $\pi(x)p(x, y) = \pi(y)p(y, x)$ for all x, y for reversibility) also make sense for infinite measures π.

▷ **Exercise 9.**

(a)$^\bullet$ Show that every Markov chain (V, P) with reversible measure π can be represented as biased random walk on the edge-weighted graph with symmetric weights (a.k.a. conductances) $c(x, y) = \pi(x)p(x, y)$. The reversible measure then, similarly to the degree in the unweighted case, is $\pi(x) = \sum_y c(x, y)$.

(b)$^\bullet$ Show that a Markov chain (V, P) has a reversible measure if and only if for all oriented cycles $x_0, x_1, \ldots, x_n = x_0$, we have $\prod_{i=0}^{n-1} p(x_i, x_{i+1}) = \prod_{i=0}^{n-1} p(x_{i+1}, x_i)$.

(c)$^\bullet$ In particular, biased random walk (probability $p > 1/2$ to the right) on the n-cycle $\mathbb{Z}$ (mod n) has a unique stationary distribution that is not reversible, while on $\mathbb{Z}$ it has two very different stationary measures (not related by a global constant factor): one is reversible, the other is not.

It makes sense to do "discrete potential analysis" on graphs and Markov chains. To start with, for any measure $\pi : V \longrightarrow [0, \infty)$, one can define the inner product $(f, g)_\pi := \sum_{x \in V} f(x)g(x)$, and the corresponding Hilbert space $L^2(V, \pi)$.

▷ **Exercise 10.**$^{\bullet\bullet}$ Show that the Markov operator P is self-adjoint with respect to the inner product $(\cdot, \cdot)_\pi$ if and only if π is a reversible measure for the Markov chain.

▷ **Exercise 11.** Let (V, P) be a Markov chain.

(a)• Green's function associated to P is

$$\mathsf{G}(x,y) := \mathbf{E}_x\big[\,\text{number of times the chain visits } y\,\big] = \sum_{n\geq 0} p_n(x,y) \;\text{ for } x,y \in V.$$

Show that if P is transient (every state is transient), then $\mathsf{G}(x,y) < \infty$ for every x, y.

(b)•• "Green's function is the inverse of the Laplacian" Let (V, P) be a transient Markov chain with a stationary measure π and associated Laplacian $\Delta = I - P$. Assume that the normalized Green's function $y \mapsto \mathsf{G}(x,y)/\pi_y$ is in $L^2(V,\pi)$. Let $f : V \longrightarrow \mathbb{R}$ be an arbitrary function in $L^2(V,\pi)$. Solve the equation $\Delta u = f$.

Let $\{Z_n\}_{n\geq 0}$ be iid standard normal variables, and $\{c_n\}_{n\geq 0}$ deterministic constants. Let us just accept (though it is not very difficult to prove, using Cauchy's convergence criterion and Kolmogorov's inequality) that if $\sum_{n\geq 0} c_n^2 < \infty$, then $\sum_{n\geq 0} c_n Z_n$ is almost surely convergent, with the limit having normal distribution with mean 0, variance $\sum_{n\geq 0} c_n^2$. Using this, here is one way to define 1D Brownian motion:

▷ **Exercise 12 (The Fourier expansion of Brownian motion).** Let Z_n be iid standard normal variables, $n = 0, 1, \ldots$, and

$$B(t) := \frac{t}{\sqrt{\pi}} \cdot Z_0 + \sqrt{\frac{2}{\pi}} \sum_{m=1}^{\infty} \frac{\sin(mt)}{m}\, Z_m.$$

Prove that:

(a)• For any $t \geq 0$ fixed, $B(t)$ is almost surely finite.

(b)• Almost surely, $B(t)$ is finite for all $t \geq 0$.

(c)• $\mathrm{Cov}(B(s), B(t)) = \min\{s, t\}$.

(d)•• Can you show that $B(t)$ is a.s. continuous?

Hints:

- $\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$.
- Taking the Fourier transform of the right hand side below, show and then use:

$$\sum_{k=1}^{\infty} \frac{\cos(kx)}{k^2} = \frac{3x^2 - 6\pi x + 2\pi^2}{12}, \qquad 0 \leq x \leq 2\pi.$$

Recall that a map $f : D_1 \longrightarrow D_2$ between open domains $D_i \subseteq \mathbb{C}$ is called conformal if $f'(z)$ exists and is non-zero for every $z \in D_1$, and f is a bijection from D_1 to D_2.

We also discussed briefly that the trajectory, hence the hitting measure of 2D Brownian motion (which is just independent 1D Brownian motions in the two coordinates) is conformally invariant. A closely related fact is that the Laplacian $\Delta = \partial^2_{xx} + \partial^2_{yy}$ is conformally invariant: it is local (it is a differential operator), linear, and rotationally invariant (frame independent). Here is a cool exercise on how to use these facts.

▷ **Exercise 13.**

(a)• Show that the map $J : z \mapsto \frac{1}{2}(z + \frac{1}{z})$ is conformal from $\mathbb{U} \setminus \{0\}$, the open unit disk with its center removed, to $\mathbb{C} \setminus [-1, 1]$. (Usually one fills in the map at $z = 0$ and gets a map to $\hat{\mathbb{C}} \setminus [-1, 1]$, where $\hat{\mathbb{C}}$ is the Riemann sphere, but you can ignore that now.) What happens to the unit circle $\partial\mathbb{U}$ under J?

(b)• For 2D Brownian motion in $\mathbb{U}$ started from 0, the hitting measure on the unit circle $\partial\mathbb{U}$ is of course uniform, by rotational invariance. Use part (a) and conformal invariance to calculate the hitting measure on $[-1, 1]$ for Brownian motion started infinitely far in $\mathbb{C}$: just a density transformation exercise. (This is one reason why lightning rods work.)

(c)• Consider the circle in $\mathbb{C}$ with center $-\epsilon + i\delta$, radius $1 + \delta$, satisfying $(1+\epsilon)^2 + \delta^2 = (1+\delta)^2$, small $\epsilon, \delta > 0$, say $\delta = 0.05$. What is the image of this under J? Does it remind you of anything? (Recalling that $\Delta = \mathrm{div}\nabla$, and hence a divergent-free gradient vector field remains divergent-free under conformal maps, this image is one reason why airplanes work.)