Gentle Statistical Mechanics — Fourth HW problem set

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▷ **Exercise 1.** For spin configuration $\sigma : V \longrightarrow \{\pm 1\}$ on a finite graph $G(V,E)$, consider some energy functional $H(\sigma)$. Use Lagrange multipliers to answer these:

(a)$^\bullet$ For any possible energy level $E \in \mathbb{R}$, among all probability measures μ on $\{\pm 1\}^{V(G)}$ with $\mathbf{E}_\mu[\,H(\sigma)\,] = E$, which are the measures maximizing the entropy $\mathsf{Ent}(\mu) := -\sum_\omega \mu(\omega)\log\mu(\omega)$?

(b)$^\bullet$ Without fixing the energy level, what are measures maximizing $\lambda\mathbf{E}_\mu[H] + \mathsf{Ent}(\mu)$ for $\lambda \in \mathbb{R}$ fixed?

As in class, the **Ising model** on a finite graph $G(V,E)$ is the random spin configuration $\sigma : V \longrightarrow \{\pm 1\}$ defined as follows. Given an external magnetic field $h \in \mathbb{R}$, the Hamiltonian is

$$H_h(\sigma) := -h \sum_{x \in V(G)} \sigma(x) - \sum_{(x,y)\in E(G)} \sigma(x)\sigma(y)\,,$$

and then the measure, at inverse temperature $\beta = 1/T \geq 0$, is

$$\mathbf{P}_{\beta,h}[\sigma] := \frac{\exp(-\beta H_h(\sigma))}{Z_{\beta,h}}\,, \quad \text{where} \quad Z_{\beta,h} := \sum_\sigma \exp(-\beta H_h(\sigma))\,.$$

▷ **Exercise 2.** The partition function $Z_{\beta,h}$ contains a lot of information about the model:

(a)$^\bullet$ Show that the **expected total energy** is

$$\mathbf{E}_{\beta,h}[\,H\,] = -\frac{\partial}{\partial\beta}\ln Z_{\beta,h}\,, \text{ with variance } \mathrm{Var}_{\beta,h}[H] = -\frac{\partial}{\partial\beta}\mathbf{E}_{\beta,h}[\,H\,]\,.$$

(b)$^\bullet$ The **average free energy** or **pressure** is defined by $f(\beta,h) := (\beta|V|)^{-1}\ln Z_{\beta,h}$. Show that for the **average total magnetization** $M(\sigma) := |V|^{-1}\sum_{x\in V}\sigma(x)$, we have

$$m(\beta,h) := \mathbf{E}_{\beta,h}[\,M\,] = \frac{\partial}{\partial h} f(\beta,h)\,.$$

(c)$^\bullet$ The **susceptibility** of the total magnetization to a change in the external magnetic field is

$$\chi(\beta,h) := \frac{1}{\beta}\frac{\partial}{\partial h} m(\beta,h) = \frac{1}{\beta}\frac{\partial^2}{\partial h^2} f(\beta,h)\,.$$

Relate this quantity to $\mathrm{Var}_{\beta,h}[M]$. Deduce that $f(\beta,h)$ is convex in h.

▷ **Exercise 3.**$^{\bullet\bullet}$ On a sequence of transitive finite graphs G_n, using the previous exercise and maybe accepting some extra assumptions, can you deduce from the Long-Range Order (LRO) condition

$$\mathrm{Var}_{\beta,h=0}[M_n] \asymp 1$$

at some shared $\beta > 0$ and $h = 0$, the Spontaneous Symmetry Breaking (SSB) property

$$\lim_{h\to 0+}\lim_{n\to\infty}\mathbf{E}_{\beta,h}[\,M_n\,] > 0 > \lim_{h\to 0-}\lim_{n\to\infty}\mathbf{E}_{\beta,h}[\,M_n\,]\,?$$

▷ **Exercise 4.** Consider the Ising model on an interval, $\{\sigma_i : i = -n, \dots, n-1, n\}$, with no boundary condition, at any inverse temperature $\beta \in [0, \infty)$.

(a)$^\bullet$ Show that $\{\sigma_i : i = -n, \dots, n-1, n\}$ is the trajectory of a stationary irreducible Markov chain on $\{-, +\}$.

(b)$^{\bullet\bullet}$ Show that $S_n := \sum_{i=-n}^{n} \sigma_i$ has $\mathrm{Var}[S_n] \sim C_\beta n$, as $n \to \infty$, for some $C_\beta \in (0, \infty)$.
That is, in dimension 1, there is no long range order for any $\beta < \infty$.

The next exercise is basically repeated from a class several weeks ago, when we talked about the Harris-FKG inequality. Stochastic domination will be a key notion in the study of the Ising model.

▷ **Exercise 5.**$^\bullet$ Let $(\Omega, \le)$ be a partially ordered set; think of $(\mathbb{R}, \le)$ or $(\{0,1\}^{\mathbb{Z}^d}, \subseteq)$. Let π a probability measure on $\Omega \times \Omega$ with the property that $\pi\big(\{(x,y) \in \Omega \times \Omega : x \le y\}\big) = 1$. Let the first marginal of π be $\mu(A) := \pi(A \times \Omega)$ and the second marginal be $\nu(A) := \pi(\Omega \times A)$ for any event $A \subseteq \Omega$. So, π is a **monotone coupling** of μ and ν. Show that ν **stochastically dominates** μ: for any increasing (upward closed) event A, we have $\mu(A) \le \nu(A)$. (In class we talked about stochastic domination and coupling for random variables, not probability measures, but if X is a random variable with values in Ω, then there is an associated measure $\mu(A) := \mathbf{P}[\,X \in A\,]$, and then everything is the same.)

By the way, Strassen's theorem says that the converse also holds: if ν stochastically dominates μ, then a monotone coupling does exist.

▷ **Exercise 6.** By constructing monotone couplings, show the following:

(a)$^\bullet$ If $n \le m$, then $\mathsf{Binom}(m,p)$ stochastically dominates $\mathsf{Binom}(n,p)$.

(b)$^\bullet$ If $0 \le p \le q \le 1$, then $\mathsf{Binom}(n,q)$ stochastically dominates $\mathsf{Binom}(n,p)$.

(c)$^\bullet$ If S_n and $\tilde{S}_n$, $n \ge 0$ are nearest neighbour (± 1) random walks on $\mathbb{Z}$ started at $S_0 = \tilde{S}_0 = 1$, with jump distributions

$$\mathbf{P}\big[\,S_{n+1} = S_n + 1 \,\big|\, S_n\,\big] = p \le q = \mathbf{P}\big[\,\tilde{S}_{n+1} = \tilde{S}_n + 1 \,\big|\, \tilde{S}_n\,\big],$$

and $\tau := \inf\{n : S_n = 0\}$ and $\tilde{\tau} := \inf\{n : \tilde{S}_n = 0\}$, then $\tilde{\tau}$ stochastically dominates τ.