Applications of Stochastics — Exercise sheet 6

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Let $\xi_1, \xi_2, \dots$ be the i.i.d. lifetimes in a renewal process, with distribution function $F(s) = \mathbf{P}[\, \xi \le s \,]$ and mean $\mathbf{E}\xi = \mu \in (0, \infty)$. Then $T_k := \sum_{i=1}^k \xi_i$ are the renewal times, $N_t := \min\{k : T_k \ge t\}$, and $U(t) := \mathbf{E}N_t$. The excess lifetime (or overshoot) is $\gamma_t := T_{N_t} - t$, the current lifetime is $\delta_t := t - T_{N_{t-1}}$, and the total lifetime is $\beta_t := \gamma_t + \delta_t$.

▷ **Exercise 1.** Find the renewal equation $H(t) = h(t) + H * F(t)$ for $H(t) := \mathbf{P}[\, \beta_t > x \,]$, where $x \ge 0$ is fixed arbitrarily.

If you did the previous exercise correctly, you understand why we are interested in the next one:

▷ **Exercise 2.** Show that, for any distribution function $F(t)$,

$$\int_0^\infty 1 - F(\max\{x, t\})\, dt = \int_x^\infty s\, dF(s)\,.$$

▷ **Exercise 3.*** Consider the so-called stationary renewal process, i.e., the delayed process with delay T_0 that has distribution function

$$\mathbf{P}[\, T_0 \le s \,] = G(s) := \frac{1}{\mu} \int_0^s 1 - F(y)\, dy\,.$$

Show that, in this G-delayed process, the overshoot γ_t^G has distribution G, for any $t \ge 0$. (Hint: first, write down a renewal type equation by conditioning on T_0. Then, use the solution from Durrett's Theorem 4.4.4 for $H_x(t) = \mathbf{P}[\, \gamma_t > x \,]$ in the undelayed process.)

▷ **Exercise 4.** Show that the stationary renewal process for $\xi_i \sim \mathsf{Expon}(\lambda)$, defined using the delay from the previous exercise, has the same distribution as the original undelayed one.

▷ **Exercise 5.** If X is a non-negative random variable with finite expectation, then its **size-biased version** $\widehat{X}$ is defined by $\mathbf{P}[\widehat{X} \in A] = \mathbf{E}[\, X\, \mathbf{1}_{\{X \in A\}} \,]/\mathbf{E}X$.

- **(a)** Show that if X has distribution function $F(t)$, then $\widehat{X}$ has distribution function $\frac{1}{\mathbf{E}X} \int_0^t s\, dF(s)$.
- **(b)** Show that the size-biased version of $\mathsf{Poi}(\lambda)$ is just $\mathsf{Poi}(\lambda) + 1$.
- **(c)** Show that the size-biased version of $\mathsf{Expon}(\lambda)$ is the sum of two independent $\mathsf{Expon}(\lambda)$'s.
- **(d)*** Take Poisson point process of intensity λ on $\mathbb{R}$. Condition on the interval $(-\epsilon, \epsilon)$ to contain at least one arrival. As $\epsilon \to 0$, what is the point process we obtain in the limit? What does this have to do with parts (a) and (b)?

▷ **Exercise 6.**

- **(a)** Using Exercises 1, 2, 5(a), and the Renewal Theorem, conclude that the limit distribution of the total lifetime β_t is the size-biased version of ξ.
- **(b)** Write a renewal equation for the current lifetime δ_t, then use the Renewal Theorem to find it has the same limit distribution as the overshoot γ_t. Are we surprised?

▷ **Exercise 7.*** Consider the random walk $S_k := \xi_1 + \cdots + \xi_k$ on the integers, with $S_0 = 1$ and i.i.d. jumps $\xi_i \sim \mathsf{Poi}(1) - 1$, and let $\tau(n) := \min\{k : S_k \geq n\}$. Recall from the martingale arguments in the Erdős-Rényi largest cluster phase transition (specifically, Exercise 4 from Sheet 3) that we were interested in the overshoot $S_{\tau(n)} - n$. Using the Renewal Theorem, find the limit distribution of this overshoot, as $n \to \infty$.

▷ **Exercise 8.** Consider the renewal process with a non-arithmetic renewal distribution with finite mean. Show that $\lim_{t\to\infty} \mathbf{P}\big[\text{number of renewals in } [0,t] \text{ is odd}\big] = 1/2$.