Applications of Stochastics — Exam questions

Gábor Pete
http://www.math.bme.hu/~gabor

1. The evolution of the Erdős-Rényi random graph $G(n,p)$: stochastic domination via the standard coupling. First and second moment method for the phase transition of containing a fixed subgraph. An example of a subgraph where this method fails.
2. Galton-Watson phase transition: three proofs. (1. Generating functions. 2. First and second moments, martingale convergence. 3. Exploration random walk and the integer-valued Chung-Fuchs theorem for recurrence.)
3. Large deviation bounds: Chernoff (exponential Markov and moment-generating function); the LD rate function for $\mathsf{Bernoulli}(p)$ using Stirling's formula; Azuma-Hoeffding for martingale differences.
4. Component structure of $G(n,(1+\epsilon)/n)$ for $\epsilon < 0$, $\epsilon = 0$, $\epsilon > 0$, using the exploration random walk.
5. Barabási-Albert preferential attachment graphs. Degree distribution: system of difference equations for the expectations, and concentration via Azuma-Hoeffding.
6. Beyond degree distribution: Clustering coefficient, PageRank.
7. First and Second Borel-Cantelli Lemmas, and the Strong Law of Large Numbers under a finite fourth moment assumption. Application of SLLN to renewal processes. The Elementary Renewal Theorem.
8. Delayed renewal processes, special delay to get a stationary process. Blackwell's Renewal Theorem: proof only for non-arithmetic ξ with finite mean $\mu < \infty$, using coupling with the stationary process.
9. Solving renewal equations. The Renewal Theorem. The renewal paradox.
10. Percolation theory basics: the equivalence of different definitions of p_c, Harris-FKG inequality, $p_c(\mathbb{Z}) = 1$, $p_c(\mathbb{T}_d) = 1/(d-1)$, $1/3 \le p_c(\mathbb{Z}^2, \mathsf{bond}) \le 2/3$.
11. Ising model, spatial Markov property, stating the entropy maximization principle, basic properties of entropy.
12. The phase transition in the Curie-Weiss model.
13. A Q/Q/1 queuing model blows up iff $\lambda > \mu$. The limiting waiting time in queue has the same distribution as the maximum of the random walk. Little's law.
14. Embedded Markov chains and explicit calculations in M/G/1 systems. Pollaczek-Khinchin formula.
15. Fluid queuing models: the PDE for first order homogeneous infinite buffer models. Some conditions for the existence of stationary solutions, and how to find them.
16. The copula of a joint distribution. Sklar's theorem. Fréchet-Hoeffding copula bounds, comonotone random variables. Using Gaussian copulas in default correlation.