Applications of Stochastics — Exercise sheet 2

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Bonus exercises are marked with a star. They can be handed in for extra points.

▷ **Exercise 1.** Let $\nu_1, \nu_2, \ldots$ be positive reals satisfying the recursion

$$\nu_{t+1} = \left(1 - \frac{\alpha_t}{t}\right)\nu_t + \frac{\beta_t}{t},$$

where α_t, β_t are positive reals converging to some positive α and β, respectively. Find $\lim_{t\to\infty} \nu_t$. (In class, we did the special case when $\alpha_t = \alpha$, $\beta_t = \beta$.)

▷ **Exercise 2.**

(a) When P is the Markov transition matrix for any finite directed graph $G(V,E)$, show that $\|Pf\|_\infty \leq \|f\|_\infty$ holds for any $f : V \longrightarrow \mathbb{R}$.

(b) Deduce from part (a) that all the eigenvalues of P have absolute value at most 1.

(c) Let A be the symmetric $n \times n$ adjacency matrix of an undirected finite graph on the vertex set $\{1, \ldots, n\}$. Let D be the diagonal matrix formed by the degrees $\deg(i)$, and note that it is clear what $D^{-1/2}$ means. Show that $B = D^{-1/2} A D^{-1/2}$ is a symmetric matrix that is conjugate to P. Deduce from part (b) that all the eigenvalues of B are real, are between 1 and -1, and that the vector $\left(\sqrt{\deg(i)}\right)_{1\leq i\leq n}$ is an eigenvector for the eigenvalue 1.

Remark. Graph theorists prefer B to P because it is symmetric, and to A because it is normalized to have spectrum between 1 and -1.

▷ **Exercise 3.** In Google's **PageRank**, the iteration $\overline{x}_{t+1} := \alpha \overline{x}_t P + (1-\alpha)\mathbf{1}$ is used, with some $\alpha \in (0,1)$. Show that, for any starting vector $\overline{x}_0$, the sequence $\overline{x}_t$ converges to $(1-\alpha)\mathbf{1}\,(I - \alpha P)^{-1}$. (Hint: use the Banach fixed point theorem, with an appropriate notion of distance; see part (a) of Exercise 2. Also, note that part (b) implies that $I - \alpha P$ is invertible for any $\alpha \in (0,1)$.)

▷ **Exercise 4.** Consider the undirected graph on the vertex set $\{1,2,3,4\}$, where $1,2,3$ form a triangle, and 1 and 4 are also connected by an edge.

(a) Calculate the eigenvector importance from part (a) of the previous exercise.

(b) Calculate the PageRank scores from part (b) of the previous exercise, for several values of α.

You are welcome to use Mathematica or other software.

▷ **Exercise 5.** Recall that we defined the **clustering coefficient** of an undirected graph as

$$\mathsf{CC} := \frac{\#\text{ paths of length 2 with endpoints connected by an edge}}{\#\text{ paths of length 2}}.$$

With n vertices and $10n$ edges, find a graph with small CC, and another one with large CC.

The following exercise appeared in class verbatim, but I put it here for those who were not present.

▷ **Exercise 6.** Let $\xi_1, \xi_2, \ldots$ be the i.i.d. lifetimes of the light bulbs, with $\mathbf{E}\xi_i = \mu \in (0, \infty]$, and we have a janitor who visits the corridor at times given by a Poisson process with intensity λ, and if he sees that the bulb is dead, he replaces it by a new one. Thus the times $\tau_1, \tau_2, \ldots$ passing between the death of a light bulb and the next visit of the janitor are thus i.i.d. $\mathsf{Expon}(\lambda)$ variables.

(a) At what rate are bulbs replaced?

(b) What is the almost sure limiting fraction of visits by the janitor on which the bulb is working?

(c) What is the limiting fraction of time that the light works?

▷ **Exercise 7.** Mr Smith likes the brand UniCar. These cars break down after a uniform $\mathsf{Uni}[0,2]$ years of use, independently of everything. Mr Smith wants to replace each of his old cars after a fixed T years of use, or the time of breakdown, whichever happens earlier. When a car breaks down, there is a cost of USD 1000 for towing it from the road and getting rid of it, and a new car costs USD 12000. If he replaces a car when it still works, he gets a discount at the store for the old car, so the new car costs only USD 10000 (and there is no extra cost of getting rid of the old car). How should Mr Smith choose T to optimize his spendings on the long run?

▷ **Exercise 8.** If X is a non-negative random variable with finite expectation, then its **size-biased version** $\widehat{X}$ is defined by

$$\mathbf{P}[\widehat{X} \in A] = \frac{\mathbf{E}[\, X\, \mathbf{1}_{\{X \in A\}}\,]}{\mathbf{E}[\, X\,]}.$$

If this looks incomprehensible to you, think of just two special cases: when X is discrete, with possible values $\{x_k\}_{k \geq 1}$, then $\mathbf{P}[\widehat{X} = x_k] = x_k \mathbf{P}[X = x_k]/\mathbf{E}X$; when X has a density function $f_X(x)$, then $\widehat{X}$ has density $f_{\widehat{X}}(x) = x f(x)/\mathbf{E}X$.

(a) Show that the size-biased version of $\mathsf{Poi}(\lambda)$ is just $\mathsf{Poi}(\lambda) + 1$.

(b) Show that the size-biased version of $\mathsf{Expon}(\lambda)$ is the sum of two independent $\mathsf{Expon}(\lambda)$'s.

(c) * Take a Poisson point process of intensity λ on $\mathbb{R}$. Condition on the interval $(-\epsilon, \epsilon)$ to contain at least one arrival. As $\epsilon \to 0$, what is the point process we obtain in the limit? What does this have to do with parts (a) and (b)?