Applications of Stochastics — Exercise sheet 3

Gábor Pete
http://www.math.bme.hu/~gabor

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Let $\xi_1, \xi_2, \dots$ be the i.i.d. lifetimes in a renewal process, with non-arithmetic distribution function $F(s) = \mathbf{P}[\,\xi \le s\,]$ and mean $\mathbf{E}\xi = \mu \in (0,\infty)$. Then $T_k := \sum_{i=1}^k \xi_i$ are the renewal times, $N_t := \min\{k : T_k \ge t\}$, and $U(t) := \mathbf{E}N_t$. The excess lifetime (or overshoot) is $\gamma_t := T_{N_t} - t$, the current lifetime is $\delta_t := t - T_{N_t - 1}$, and the total lifetime is $\beta_t := \gamma_t + \delta_t$.

▷ **Exercise 1.**

(a) Find the renewal equation $H(t) = h(t) + H * F(t)$ for $H(t) := \mathbf{P}[\,\beta_t > x\,]$, where $x \ge 0$ is fixed arbitrarily. (We actually did this in class.)

(b) Find the renewal equation for $H(t) := \mathbf{P}[\,\gamma_t > x\,]$.

(c) Using the Renewal Theorem, find the limit distributions of β_t and γ_t as $t \to \infty$.

If you did the previous exercise correctly, you understand why we are interested in the next one:

▷ **Exercise 2.**

(a) Show that, for any distribution function $F(t)$,

$$\int_0^\infty 1 - F(\max\{x,t\})\, dt = \int_x^\infty s\, dF(s)\,.$$

(b) Show that if X has distribution function $F(t)$, then the size-biased version $\widehat{X}$ has distribution function $\frac{1}{\mathbf{E}X} \int_0^t s\, dF(s)$.

From the previous two exercises, conclude the following:

▷ **Exercise 3.**

(a) The limit distribution of the total lifetime β_t is the size-biased version of ξ.

(b) The limit distribution of the overshoot γ_t is the size-biased version $\widehat{\xi}$ multiplied with an independent $\mathsf{Unif}[0,1]$ variable.

▷ **Exercise 4.**

(a) Recall (or prove now again) that if $\xi_1 + \eta_1 + \xi_2 + \eta_2 + \dots$ is an alternating renewal process with expectations $\mathbf{E}\xi_i = \mu \in (0,\infty)$ and $\mathbf{E}\eta_i = \lambda \in (0,\infty)$, then the asymptotic proportion of time spent in ξ-intervals is $\mu/(\mu+\lambda)$.

(b) A harder, local version can be proved using an appropriate renewal equation and the Renewal Theorem: if the distribution of the independent sum $\xi_i + \eta_i$ is non-arithmetic, then the probability that moment t is in a ξ-interval converges to $\mu/(\mu+\lambda)$ as $t \to \infty$.

(c) As a special case, show that in a renewal process with a non-arithmetic renewal distribution with finite mean, $\lim_{t\to\infty} \mathbf{P}\big[\,\text{number of renewals in } [0,t] \text{ is odd}\,\big] = 1/2$.

(d)* Does the last conclusion remain true if the renewal time has infinite mean?

▷ **Exercise 5.** For iid positive variables $B_1, B_2, \dots$ with finite mean, consider $Z_n := B_1 + \cdots + B_n$. Take a random point $U_n \sim \mathsf{Unif}[0, Z_n]$, and let K_n be the index that satisfies $Z_{K_n - 1} \le U_n < Z_{K_n}$. Show that B_{K_n} converges in distribution to the size-biased version $\widehat{B}$.

▷ **Exercise 6.** Read the proof from Feller that $W_n \overset{d}{=} \max\{0, S_1, \dots, S_n\}$, where S_i is the random walk with increments $X_i := \mathcal{B}_{i-1} - \mathcal{A}_i$.

(a) Assume that $\mathbf{E}X_i < 0$. Then $S_{\max} = \max\{0, S_1, S_2, \dots\}$ is an almost surely finite variable, since $S_n \to -\infty$. Assume that $\mathbf{E}\big[\, e^{tX_i} \,\big] < \infty$ for some $t > 0$. Show that $\mathbf{P}\big[\, S_{\max} > m \,\big] < C \exp(-cm)$ for some $0 < c, C < \infty$. In particular, $\mathbf{E}S_{\max} < \infty$.

(b) Assuming $\mathbf{E}S_{\max} < \infty$, as in the previous item, show that $W_n/n \to 0$ almost surely. Recall from class that this implies that the a.s. limiting utilization ratio is λ/μ.

(c) Give an example of a sequence of random variables V_n on a single probability space so that they converge in distribution to an almost surely finite variable V_∞, but V_n/n does not converge almost surely to 0.

(d) * In the case $\mathbf{E}X_i < 0$, can it happen that $\mathbf{E}S_{\max} = \infty$? Can it happen that W_n/n does not converge almost surely to 0?

▷ **Exercise 7.** Consider an **M/G/1 system**: the arrival process is Markovian, with rate λ, the service is general, with rate μ. Let $H_1 := \inf\{t > 0 : Q_t^+ = 0\}$ be the length of the first busy period. Assume $\lambda \le \mu$.

Show that the busy and idle periods form an alternating renewal process. Using Exercise 4 (a) and Little's law, show that $\mathbf{E}H_1 = \frac{1}{\mu - \lambda}$.

▷ **Exercise 8.** Show that the copula $C(u_1, \dots, u_n)$ of any n-dimensional joint distribution satisfies

$$\max\left\{ 1 - n + \sum_{i=1}^{n} u_i \,,\, 0 \right\} \le C(u_1, \dots, u_n) \le \min\{u_1, \dots, u_n\}\,.$$

Show by examples that the upper bound is sharp for any $n \ge 1$, while the lower bound is sharp for $n = 1, 2$.