Diszkrét Véletlen Struktúrák, 2016 tavasz
Házi feladatok

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Bocsánat, de néha angolul, néha magyarul vannak a feladatok. Akit nagyon zavar, szóljon. A vizsgára **négy** megoldást hozzatok, de biztonságosabb, ha előbb adjátok be, mert ha hibás, akkor lehet javítani. A csillagos * példák kettőt érnek.

1 Véletlen gráfok és egyéb konstrukciók

$\triangleright$ **Exercise 1.** Let $G = (V, E)$ be a bipartite graph with n vertices and a list $S(v)$ of more than $\log_2 n$ colors associated with each vertex $v \in V$. Prove that there is a proper coloring of G assigning to each vertex v a color from its list $S(v)$.

$\triangleright$ **Exercise 2.*** Vegyünk $\mathbb{R}^n$-ben több mint Mk darab páronként merőleges egységvektort, és vetítsük le őket egy adott $\mathbb{R}^k$ altérre (mondjuk vegyük az első k koordinátájukat). Igazoljuk, hogy van olyan vetített, aminek a hossza kisebb mint $1/\sqrt{M}$. Példával mutassuk, hogy $Mk = 2^r$ darab vektor esetén ez még nem igaz.

$\triangleright$ **Exercise 3.** For any monotone increasing event $\mathcal{A}$ on n bits, we define $p_t^{\mathcal{A}} := \inf \big\{ p : \mathbf{P}_p[\,\mathcal{A}\,] \geq t \big\}$.

Prove the Bollobás-Thomason **threshold** theorem: for any sequence of monotone increasing events $\mathcal{A} = \mathcal{A}_n$ and any ϵ there is $C_\epsilon < \infty$ such that $\big|p_{1-\epsilon}^{\mathcal{A}}(n) - p_\epsilon^{\mathcal{A}}(n)\big| < C_\epsilon \left(p_\epsilon^{\mathcal{A}}(n) \wedge (1 - p_{1-\epsilon}^{\mathcal{A}}(n)) \right)$. (Hint: take many independent copies of low density to get success with good probability at a larger density.)

$\triangleright$ **Exercise 4.** Find the order of magnitude of the threshold function $p_{1/2}(n)$ for the random graph $G(n, p)$ containing a copy of the cycle C_4. (Hint: compute the first and second moments of the number of C_4 copies, and use Chebyshev's inequality.)

$\triangleright$ **Exercise 5.** Igazoljuk a Lovász Lokális Lemma segítségével, hogy $R(k, 4) \geq k^{5/2+o(1)}$.

$\triangleright$ **Exercise 6.** Igazoljuk, hogy minden $\epsilon > 0$-hoz létezik $\delta > 0$, hogy tetszőleges n-re a $G_{n,n,d}$ uniform páros d-reguláris véletlen gráf Cheeger-konstansa legalább $1 - \epsilon$ valószínűséggel δ. (Tipp: ki kell számolni az olyan $(S, T) \subset [n] \times [n]$ részhalmaz-párok várható számát, S a jobb, T a bal oldalon, $|S| < n/2$ és $|T| < (1 + \gamma)|S|$, ahol S összes szomszédja, d független teljes párosítást követve, T-ben van. Ha γ elég kicsi, akkor ez kicsi lesz, így annak a valószínűsége, hogy van ilyen, 0-hoz tart. Ebből könnyen következik, hogy $G_{n,n,d}$ nagy valószínűséggel expander.)

$\triangleright$ **Exercise 7.*** A graph is called t-tough if for every $m \geq 2$ we need to delete at least tm vertices to get at least m connected components. For instance, any graph possessing a Hamiltonian cycle is 1-tough. On the other hand, t-tough graphs have independence number $\alpha(G) \leq n/(t + 1)$, since otherwise we could delete less than $tn/(t + 1)$ vertices and get more than $n/(t + 1)$ isolated vertices.

Show that for every t there is a $d = d(t)$ such that the d-regular uniform random graph $G_{2n,d}$ is t-tough with large probability.

2 Gráfok, csoportok geometriája

▷ **Exercise 8.*** Consider the standard hexagonal lattice. Show that if you are given a bound $B < \infty$, and can group the hexagons into countries, each being a connected set of at most B hexagons, then it is not possible to have at least 7 neighbours for each country.

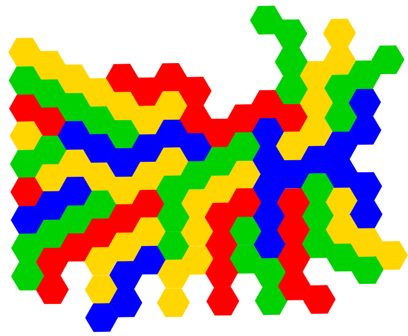

Figure 1: Trying to create at least 7 neighbours for each country.

▷ **Exercise 9.** Recall that being non-amenable means satisfying the strong isoperimetric inequality IP_∞.

(a) Show that a bounded degree tree without leaves is amenable iff there is no bound on the length of "hanging chains", i.e., chains of vertices with degree 2. (Consequently, for trees, $IP_{1+\epsilon}$ implies IP_∞.)

(b) Give an example of a bounded degree tree of exponential volume growth that satisfies no $IP_{1+\epsilon}$, recurrent for the simple random walk on it, and has $p_c = 1$ for Bernoulli percolation.

▷ **Exercise 10.*** Show that a bounded degree graph $G(V, E)$ is nonamenable if and only if it has a wobbling paradoxical decomposition: two injective maps $\alpha, \beta : V \longrightarrow V$ such that $\alpha(V) \sqcup \beta(V) = V$ is a disjoint union, and both maps are at a bounded distance from the identity, or wobbling: $\sup_{x \in V} d(x, \alpha(x)) < \infty$. (Hint: State and use the locally finite infinite bipartite graph version of the Hall marriage theorem, called the Hall-Rado theorem.)

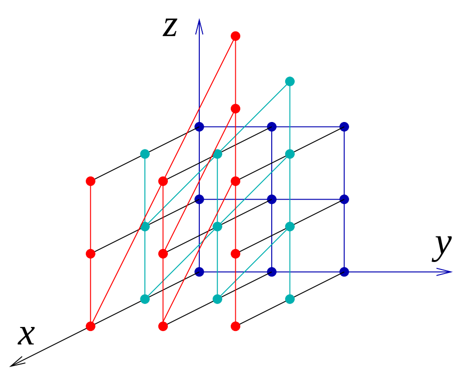

Figure 2: The Cayley graph of the Heisenberg group with generators X, Y, Z.

The **3-dimensional discrete Heisenberg group** is the matrix group

$$H_3(\mathbb{Z}) = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} : x, y, z \in \mathbb{Z} \right\}.$$

If we denote by X, Y, Z the matrices given by the three permutations of the entries $1, 0, 0$ for x, y, z, then $H_3(\mathbb{Z})$ is given by the presentation

$$\big\langle X, Y, Z \,\big|\, [X, Z] = 1, [Y, Z] = 1, [X, Y] = Z \big\rangle.$$

▷ **Exercise 11.** Show that the discrete Heisenberg group has 4-dimensional volume growth.

▷ **Exercise 12.**

(a) Show that the Diestel-Leader graph $\mathsf{DL}(k, \ell)$ is amenable iff $k = \ell$.

(b) Show that the Cayley graph of the lamplighter group $\Gamma = \mathbb{Z}_2 \wr \mathbb{Z}$ with generating set $S = \{\mathsf{R}, \mathsf{Rs}, \mathsf{L}, \mathsf{sL}\}$ is the Diestel-Leader graph $\mathsf{DL}(2, 2)$. How can we obtain $\mathsf{DL}(p, p)$ from $\mathbb{Z}_p \wr \mathbb{Z}$?

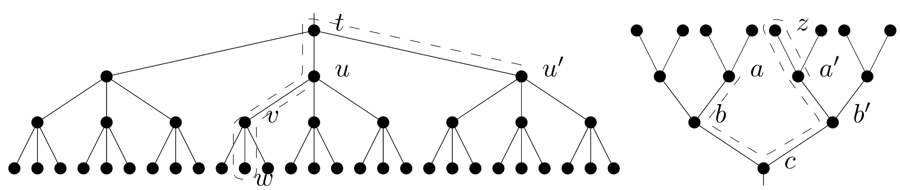

Figure 3: The Diestel-Leader graph $\mathsf{DL}(3, 2)$, with a path: (u, a), (v, b), (w, c), (v, b'), (u, a'), (t, z), (u', a').

▷ **Exercise 13.** Show that amenable transitive graphs are unimodular (that is, they satisfy the Mass Transport Principle).

3 Gráfok spektruma, lokalitás, bolyongások

▷ **Exercise 14.** *

(a) Show that the Markov operator on the d-regular tree $\mathbb{T}_d$ with $d \geq 2$ (i.e., including $\mathbb{Z}$) has no eigenvectors $\lambda f = Pf$ with $f \in \ell^2(\mathbb{T}_d)$, for any $\lambda \in \mathbb{R}$. (Hint: assuming there is one, show that there would also be one whose values depend only on the distance from the root; then exclude this by direct computation.)

(b) Show that the quasi-transitive tree T that has degree 3 and degree 2 vertices alternately does have an $\ell^2(T)$-eigenvector, with eigenvalue 0.

▷ **Exercise 15** (The spectral measure of $\mathbb{Z}$)**.** Show that for the SRW Markov operator on $\mathbb{Z}$, the Kesten spectral measure is $d\sigma_{x,x}(t) = \frac{1}{\pi\sqrt{1-t^2}} \mathbf{1}_{[-1,1]}(t)\, dt$. (Hint: you could do this in at least two ways: either from the spectrum of the cycle C_n, or from computing return probabilities and moments explicitly, and arguing that the spectral measure is determined by its moments.)

▷ **Exercise 16.** Ha korlátos fokú gráfokra egy $G \mapsto \phi(G)$ gráf-paraméter folytonos a Benjamini-Schramm lokális topológiában, azaz $G_n \xrightarrow{\text{BSch}} (\mathcal{G}, o)$ esetén $\phi(G_n)$ konvergál, akkor az lokálisan tesztelhető: minden $D < \infty$ és $\epsilon > 0$ esetén létezik $N(\epsilon, D)$ és $K(\epsilon, D)$ végesek, hogy ha G egy tetszőleges véges D max-fokú gráf legalább N csúcson, akkor K darab uniform véletlen csúcsnak megnézve a K sugarú környezetét G-ben, ezen környezetek alapján (ami csak korlátos sok infó) tudunk egy ϕ^* tippet mondani, ami a $\phi(G)$-nek ϵ sugarú környezetében lesz legalább $1 - \epsilon$ valószínűséggel.

▷ **Exercise 17.** Legyen $\pi(\cdot)$ reverzibilis valszín mérték a $p(\cdot, \cdot)$ Markov lánchoz a V állapottéren, és

$$d_\infty(n) := \sup_{x \in V} \left\| \frac{p_n(x, \cdot)}{\pi(\cdot)} - 1 \right\|_\infty .$$

Igazoljuk, hogy

$$d_\infty(n+m) \le d_\infty(n)\, d_\infty(m)\,.$$

▷ **Exercise 18.** Show that if the chain (V, P) is transitive, then

$$4\, d_{\mathrm{TV}}\big(p_t(x,\cdot), \pi(\cdot)\big)^2 \le \left\| \frac{p_t(x,\cdot)}{\pi(\cdot)} - 1 \right\|_2^2 = \sum_{i=2}^{n} \lambda_i^{2t}\,.$$

For instance, assuming the spectrum of the lazy SRW on the hypercube $\{0,1\}^k$ from class, deduce the bound $d\big(1/2\, k \ln k + c\, k\big) \le e^{-2c}/2$ for $c > 1$ on the TV distance. (This is sharp even regarding the constant 1/2 in front of $k \ln k$. Why? Think of the coupon collector's problem.) Also, deduce that $t_{\mathrm{mix}}^{\mathrm{TV}}(C_n) = O(n^2)$ for the n-cycle.

▷ **Exercise 19.** Why it is hard to construct large expanders:

(a) If $G' \longrightarrow G$ is a covering map of infinite graphs, then the spectral radii satisfy $\rho(G') \le \rho(G)$, i.e., the larger graph is more non-amenable. In particular, if G is an infinite k-regular graph, then $\rho(G) \ge \rho(\mathbb{T}_k) = \frac{2\sqrt{k-1}}{k}$. (The last equality we have not seen and is not an exercise now.)

(b) If $G' \longrightarrow G$ is a covering map of finite graphs, then $\lambda_2(G') \ge \lambda_2(G)$, i.e., the larger graph is a worse expander.

▷ **Exercise 20.** You may accept here that transitive expanders exist. Give a sequence of d-regular transitive graphs $G_n = (V_n, E_n)$ with $|V_n| \to \infty$ that mix rapidly, $t_{\mathrm{mix}}^{\mathrm{TV}}(1/4) = O(\log |V_n|)$, but do not form an expander sequence.

▷ **Exercise 21.** A simple version of the Tetris game (with no player): on the discrete cycle of length K, unit squares with sticky corners are falling from the sky, at places $[i, i+1]$ chosen uniformly at random ($i = 0, 1, \dots, K-1$, mod K). Let R_t be the size of the roof after t squares have fallen: those squares of the current configuration that could have been the last to fall. Show that $\lim_{t\to\infty} \mathbf{E} R_t = K/3$.

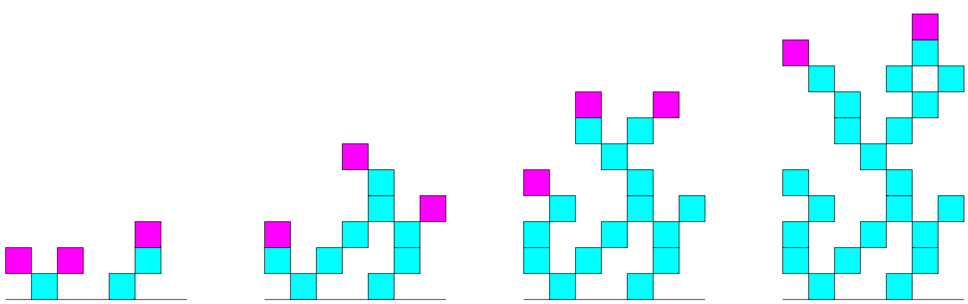

Figure 4: Sorry, this picture is on the segment, not on the cycle.

Remark. If there are two types of squares, particles and antiparticles that annihilate each other when falling on exactly on top of each other, this process is a SRW on a group, and the size of the roof has to do with the speed of the SRW. Here, for $K \ge 4$, the expected limiting size of the roof is already less than $0.32893K$, but this is far from trivial. What's the situation for $K = 3$?

The next lemma was used in the evolving sets method:

▷ **Exercise 22.** If $\mathrm{Var}[X] \ge c\, (\mathbf{E}X)^2$ then $\mathbf{E}\big[\sqrt{X}\,\big] \le (1 - c')\sqrt{\mathbf{E}X}$, where $c' > 0$ depends only on $c > 0$.

4 Perkoláció típusú folyamatok

▷ **Exercise 23.** Let $G(V, E)$ be an infinite transitive graph, and γ_n be the number of minimal edge cutsets of size n separating an origin $o \in V$ from infinity. Show, using a union bound, that if $\gamma_n < \exp(Cn)$ for some $C < \infty$, then $p_c(G) < 1$ for Bernoulli bond percolation on G.

▷ **Exercise 24.** Assume that $\pi : G' \longrightarrow G$ is a topological covering between infinite graphs, or in other words, G is a factor graph of G'. Show that $p_c(G') \le p_c(G)$.

▷ **Exercise 25.**

(a) Give a translation invariant and ergodic percolation on $\mathbb{Z}^2$ with infinitely many ∞ clusters.

(b) Give a translation invariant and ergodic percolation on $\mathbb{Z}^2$ with exactly two ∞ clusters.

▷ **Exercise 26.** As in the lecture, a furcation point of an infinite cluster is a vertex whose removal breaks the cluster into at least 3 infinite components. Show carefully the claim we used in the Burton-Keane theorem: if $\mathscr{C}_\infty$ denotes the union of all the infinite clusters in some percolation on G, and $U \subset V(G)$ is finite, then the size of $\mathscr{C}_\infty \cap \partial_V^{\mathrm{out}} U$ is at least the number of trifurcation points of $\mathscr{C}_\infty$ in U, plus 2.

▷ **Exercise 27.**

(a) In an invariant percolation process on a unimodular transitive graph G, show that almost surely the number of ends of each infinite cluster is 1 or 2 or continuum.

(b) Give an invariant percolation on a non-unimodular transitive graph that has infinite clusters with more than two but finitely many ends.

▷ **Exercise 28.** Consider the graph G with 6 vertices and 7 edges that looks like a figure 8 on a digital display. Consider the uniform measure on the 15 spanning trees of G, denoted by UST, and the uniform measure on the 7 connected subgraphs with 6 edges (one more than a spanning tree), denoted by $\mathsf{UST} + 1$. Find an explicit monotone coupling between the two measures (i.e., with $\mathsf{UST} \subset \mathsf{UST} + 1$).

Question. Is there such a monotone coupling for every finite graph?

▷ **Exercise 29.** Using Wilson's algorithm that generates a UST of a finite graph G using Loop Erased Random Walks, now for $G = K_n$, prove Cayley's formula: the number of trees on n labeled vertices is n^{n-2}.

▷ **Exercise 30.*** Using Wilson's algorithm as above, prove the following limit law for distances in a uniform random tree $\mathcal{T}_n$ on n labeled vertices: if $x \neq y$ are two uniformly chosen random vertices, then their graph distance satisfies

$$\lim_{n\to\infty} \mathbf{P}\big[\, d_{\mathcal{T}_n}(x, y) > t\sqrt{n}\,\big] = \exp(-t^2/2)\,.$$