Probability on Graphs and Groups — First problem set

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The number of dots • is the value of an exercise. Please hand in solutions, at least for 12 points, by April 5 Tue. If you need an extension for some reason, April 8 Friday is OK. I hope to have two more exercise sheets during the course.

▷ **Exercise 1.** • Consider a GW process with offspring distribution ξ, with $\mathbb{E}\xi = \mu > 1$ and $\mathbb{D}^2\xi = \sigma^2 < \infty$. Let Z_n be the size of the nth level, with $Z_0 = 1$, the root. Using the conditional variance formula $\mathbb{D}^2[Z_n] = \mathbb{E}\big[\,\mathbb{D}^2[Z_n \mid Z_{n-1}]\,\big] + \mathbb{D}^2\big[\mathbb{E}[\,Z_n \mid Z_{n-1}\,]\big]$, show that $\mathbb{E}\big[\,Z_n^2\,\big] \le C_{\mu,\sigma}(\mathbb{E}Z_n)^2$.

▷ **Exercise 2.** Let T be the Galton-Watson tree with offspring distribution $\xi \sim \mathsf{Geom}(1/2) - 1$. Draw the tree into the plane with root ρ, add an extra vertex ρ' and an edge (ρ, ρ'), and walk around the tree, starting from ρ', going through each "corner" of the tree once, through each edge twice (once on each side). At each corner visited, consider the graph distance from ρ': let this be process be $\{X_t\}_{t=0}^{2n}$, which is positive everywhere except at $t = 0, 2n$, where n is the number of vertices of the original tree T.

Figure 1: The contour walk around a tree.

(a) • Using the memoryless property of $\mathsf{Geom}(1/2)$, show that $\{X_t\}$ is SRW on $\mathbb{Z}$.

(b) • Using a martingale argument, show that $\mathbb{P}[T \text{ has height } \ge n] = 1/n$.

(c) • Using another martingale argument: what is the expected size of the nth generation conditioned on being non-empty?

(d) ••• Show that for any $\epsilon > 0$ there exists $K < \infty$ such that, conditioning T to have height at least n, with probability at least $1 - \epsilon$ the height will be at most Kn, and the total volume will be between n^2/K and Kn^2. (Hint: the typical speed of an unconditioned SRW is given by the Central Limit Theorem. But how do you compare the speed of conditioned and unconditioned trajectories?)

Recall that the α-dimensional **Hausdorff measure** of a metric space (X, d) is defined by

$$\mathcal{H}_\alpha(X) := \lim_{\epsilon \to 0} \inf \left\{ \sum_{i=1}^{\infty} \mathrm{diam}(U_i)^\alpha : \bigcup_i U_i \supset X,\ \sup_i \mathrm{diam}(U_i) < \epsilon \right\}.$$

Then $\dim_H(X) := \inf\{\alpha : \mathcal{H}_\alpha(X) = 0\}$ is the **Hausdorff dimension**, while

$$\overline{\dim}_M(X) := \overline{\lim}_{\epsilon \to 0} \frac{\log N_\epsilon(X)}{\log(1/\epsilon)} \quad \text{and} \quad \underline{\dim}_M(X) := \underline{\lim}_{\epsilon \to 0} \frac{\log N_\epsilon(X)}{\log(1/\epsilon)}$$

are the **upper and lower Minkowski dimensions**, where $N_\epsilon(X)$ is the infimum number of subsets of diameter at most $\epsilon > 0$ that are needed to cover X.

▷ **Exercise 3.** • Show that for any metric space (X, d) there is at most one $\alpha \geq 0$ such that $\mathcal{H}_\alpha(X) \in (0, \infty)$.

▷ **Exercise 4.** •• For $\alpha \in (0, \infty)$, consider $X_\alpha := \{n^{-\alpha}, n = 1, 2, \dots\} \subset [0, 1]$, with the metric inherited from $\mathbb{R}$. Find the Minkowski and Hausdorff dimensions of X_α.

▷ **Exercise 5.** Recall the metric $d(\xi, \eta) = b^{-|\xi \wedge \eta|}$, for any $b > 1$, on the boundary ∂T of a locally finite infinite tree without leaves that we considered in class. Also recall that to any $x \in V(T)$ we associated the clopen (both closed and open) set $B_x := \{\xi \in \partial T : x \in \xi\}$, and if Π is a cutset between the root and infinity, then $B_\Pi := \{B_x : x \in \Pi\}$ is obviously a cover of ∂T.

(a) • Give a countable covering by disjoint closed sets of the boundary of the binary tree that does not arise from a cutset between the root and infinity. (Note: this issue is completely neglected in the Lyons-Peres book.)

(b) •• For any countable covering $\{U_i\}$ of any ∂T with $\sum_i \mathrm{diam}(U_i)^\alpha < \infty$, and any $\epsilon > 0$, construct a finite cutset Π such that the associated covering B_Π satisfies

$$\sum_{x \in \Pi} \mathrm{diam}(B_x)^\alpha < \sum_i \mathrm{diam}(U_i)^\alpha + \epsilon.$$

(c) • Deduce from the previous item that $\mathsf{br}(T) = b^{\dim_H(\partial T)}$.

(d) • Show that $\overline{\mathsf{gr}}(T) = b^{\overline{\dim}_M(\partial T)}$ and $\underline{\mathsf{gr}}(T) = b^{\underline{\dim}_M(\partial T)}$.

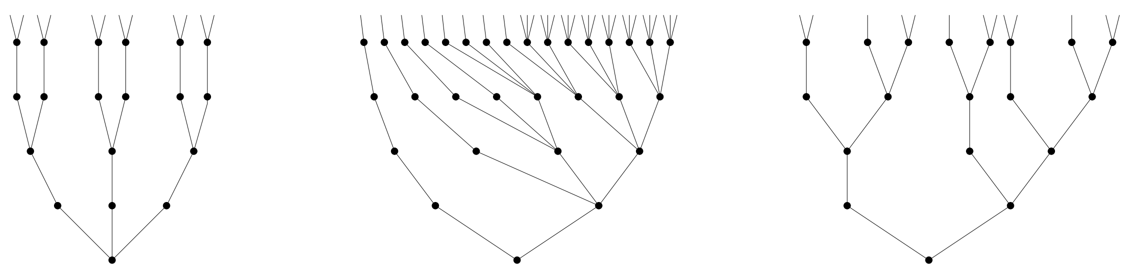

Figure 2: A quasi-transitive tree, the 3-1 tree, and the Fibonacci tree.

▷ **Exercise 6.** Find the branching number of each of the three trees on Figure 2:

(a) • A quasi-transitive tree, with degree 3 and degree 2 vertices alternating.

(b) • The so-called 3-1-tree, which has 2^n vertices on each level n, with the left 2^{n-1} vertices each having one child, the right 2^{n-1} vertices each having three children; the root has two children.

(c) •• The Fibonacci tree, which is a directed universal cover of the directed graph with vertices $\{1, 2\}$ and edges $\{(12), (21), (22)\}$. (There are two directed covers, with root either 1 or 2.)

▷ **Exercise 7.** •• Show that SRW on the 3-1 tree above is recurrent, but the Nash-Williams criterion does not work.

▷ **Exercise 8.** Consider the nearest neighbour RW on $\mathbb{Z}$ with $\mathbb{P}[\, X_{t+1} = X_t + 1\,] = p > 1/2$, and the function $h(i) := \{(1-p)/p\}^i$ for $i \in \mathbb{Z}$.

(a) • Show that $M_t := h(X_t)$ is a martingale. Using the Optional Stopping Theorem for bounded MGs, find $\mathbb{P}_i[\, \tau_a < \tau_b\,]$ for $a \leq i \leq b$.

(b) • From the previous part, find $\mathbb{P}_i[\, \tau_0 < \infty\,]$. Then give a simply reason why it has to be exactly exponentially decreasing in i.

▷ **Exercise 9.** • Show that a Markov chain (V, P) has a reversible measure if and only if for all oriented cycles $x_0, x_1, \dots, x_n = x_0$, we have $\prod_{i=0}^{n-1} p(x_i, x_{i+1}) = \prod_{i=0}^{n-1} p(x_{i+1}, x_i)$.

Recall the definition of **effective resistance** between vertices a and z in a finite graph:

$$\mathcal{R}(a \leftrightarrow z) := \frac{v(z) - v(a)}{|||\nabla v|||},$$

where v is the voltage function between a and z with $v(z) > v(a)$. We can then also define $\mathcal{R}(A \leftrightarrow Z)$ for any two disjoint subsets $A, Z \subset V(G)$, by collapsing all the points in A and Z to a single vertex a and z, respectively, keeping all the edges leaving A and Z. We can also define $\mathcal{C}(A \leftrightarrow Z) := 1/\mathcal{R}(A \leftrightarrow Z)$.

▷ **Exercise 10.** •• Show that effective resistances add up when combining networks in series, while effective conductances add up when combining networks in parallel.

▷ **Exercise 11** ("Green's function is the inverse of the Laplacian")**.** • Let (V, P) be a transient Markov chain with a stationary measure π, associated Laplacian $\Delta = I - P$, and Green's function $G(x,y) := \sum_{n=0}^{\infty} p_n(x,y)$. Assume that the function $y \mapsto G(x,y)/\pi_y$ is in $L^2(V,\pi)$. Let $f : V \longrightarrow \mathbb{R}$ be an arbitrary function in $L^2(V,\pi)$. Solve the equation $\Delta u = f$.

Recall **Thomson's principle**:

$$\mathcal{R}(a \leftrightarrow z) = \inf \Big\{ \mathcal{E}(\theta) : \theta \text{ is a flow from } a \text{ to } z \text{ with strength } |||\theta||| \geq 1 \Big\},$$

where, with a slight abuse of notation, we use the notation for Dirichlet energy also for the r-energy of a general flow: $\mathcal{E}(\theta) := \langle \theta, \theta \rangle_r$.

Note furthermore that **Dirichlet's principle** can be reformulated as follows:

$$\mathcal{R}(a \leftrightarrow z)^{-1} = \inf \Big\{ \mathcal{E}(f) : f \text{ is a function } V \longrightarrow \mathbb{R} \text{ with } f(a) \leq 0 \text{ and } f(z) \geq 1 \Big\}.$$

▷ **Exercise 12.** Using the above two principles and the methods from March 29, prove the following:

(a) • If $a = (0,0)$ and $z = (n,n)$ in the square $G = \{0, \dots, n\}^2 \subset \mathbb{Z}^2$, then $\mathcal{R}(a \leftrightarrow z) \asymp \log n$.

(b) • For the square annulus $\{-n, \dots, n\}^2 \setminus \{-k, \dots, k\}^2 \subset \mathbb{Z}^2$, if A denotes the set of inner boundary vertices (at ℓ^∞-distance k from the origin), and Z denotes the outer boundary (the vertices at ℓ^∞-distance n from the origin), then $\mathcal{R}(A \leftrightarrow Z) \asymp \log(n/k)$.

(c) •• Consider the wedge $\mathcal{W}_h := \big\{ (x,y,z) \in \mathbb{Z}^3 : x \geq 0,\ |z| \leq h(x) \big\}$, where $h(x) := (\log x)^\alpha$ for some $\alpha > 0$. For what values of α can you prove that $\mathcal{W}_h$ is recurrent? transient?

▷ **Exercise 13.** On any locally finite graph G, call $h : V(G) \longrightarrow \mathbb{R}$ **infinity-harmonic** if, for every $x \in V(G)$,

$$h(x) = \frac{1}{2} \left(\min_{y \sim x} h(y) + \max_{y \sim x} h(y) \right).$$

(The reason for this name is that these functions minimize the L^∞-norm of the gradient in a strong sense, just like usual harmonic functions minimize the L^2-norm, the Dirichlet energy.)

(a) • Show that every non-constant infinity-harmonic function h grows at least linearly in some direction: there is a sequence of vertices $(x_i)_{i \geq 0}$ such that $\liminf_{i \to \infty} h(x_i)/d(x_0, x_i) > 0$.

(b) • Design a random walk (i.e., a time-independent Markov process) $(X_t)_{t \geq 0}$ with nearest-neighbour jumps such that $M_t := h(X_t)$ is a martingale.

(c) • Show that the process $\Delta_t := \mathbb{D}[M_{t+1} \,|\, X_t]$ is almost surely non-decreasing in $t \geq 0$. Deduce that $\mathbb{D}[M_t]$ grows at least linearly in t. (I'm not claiming that on every graph, every h, for every process (X_t) such that $h(X_t)$ is a MG, there is this linear growth of the variance. But for every h there is such a process, and I bet that your construction in (b) does in fact have this property.)