Probability on Graphs and Groups — Second problem set

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The number of dots • is the value of an exercise. Please hand in solutions, at least for 12 points, by May 13, Friday. I will add a couple of more exercises next week. Some of the exercises appeared in the Stochastic Models course, as well — don't hand them in if you already did them then.

▷ **Exercise 1.** •• For what primes p, q is there a semidirect product $\mathbb{Z}_p \rtimes \mathbb{Z}_q$ that is not a direct product? (Hint: you do not have to know what exactly the group $\mathrm{Aut}(\mathbb{Z}_p)$ is; it is enough to use Cauchy's theorem on having an element of order q in any group of size n, for any prime $q|n$. But, in fact, $\mathrm{Aut}(\mathbb{Z}_p)$ is always cyclic, because there exist primitive roots modulo any prime.)

▷ **Exercise 2.** • Show that $\mathbb{Z}^2 \rtimes_M \mathbb{Z}$ with $M = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ is isomorphic to the Heisenberg group.

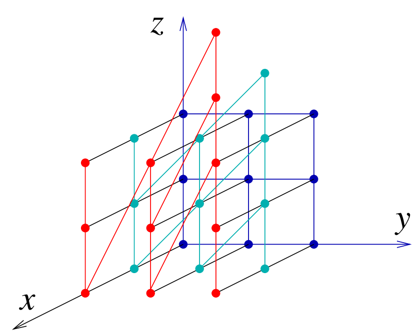

Figure 1: A Cayley graph of the Heisenberg group.

▷ **Exercise 3.** We say that a bounded degree graph $G(V, E)$ has d-dimensional volume growth if there exist $0 < c < C < \infty$ such that $cr^d < |B_r(o)| < Cr^d$ for any $o \in V$ and every large enough $r > r^*(o)$.

(a) • Show that if a group has a finitely generated Cayley graph with d-dimensional volume growth, then all its finitely generated Cayley graphs have d-dimensional volume growth.

(b) •• Show that the discrete Heisenberg group has 4-dimensional volume growth.

The **Diestel-Leader graph** $\mathsf{DL}(k, \ell)$ is the so-called **horocyclic product** of $\mathbb{T}_{k+1}$ and $\mathbb{T}_{\ell+1}$: pick an end of each tree, organize all the vertices into layers labeled by $\mathbb{Z}$ with labels tending to $+\infty$ towards that end (with the location of the zero level being arbitrary), then let $V(G)$ consist of all the pairs $(v, w) \in \mathbb{T}_{k+1} \times \mathbb{T}_{\ell+1}$ with labels $(n, -n)$ for some $n \in \mathbb{Z}$, with an edge from (v, w) to (v', w') if $(v, v') \in E(\mathbb{T}_{k+1})$ and $(w, w') \in E(\mathbb{T}_{\ell+1})$. See Figure 2.

▷ **Exercise 4.**

(a) • Show that the Cayley graph of the lamplighter group $\Gamma = \mathbb{Z}_2 \wr \mathbb{Z}$ with generating set $S = \{\mathsf{R}, \mathsf{L}, \mathsf{s}\}$ is indeed that graph that we discussed.

(b) •• Show that the Cayley graph of the lamplighter group $\Gamma = \mathbb{Z}_2 \wr \mathbb{Z}$ with generating set $S = \{\mathsf{R}, \mathsf{Rs}, \mathsf{L}, \mathsf{sL}\}$ is the Diestel-Leader graph $\mathsf{DL}(2, 2)$. How can we obtain $\mathsf{DL}(p, p)$ from $\mathbb{Z}_p \wr \mathbb{Z}$?

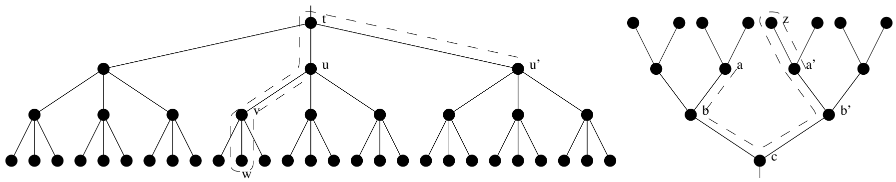

Figure 2: The Diestel-Leader graph $\mathsf{DL}(3,2)$. A path: (u,a), (v,b), (w,c), (v,b'), (u,a'), (t,z), (u',a').

▷ **Exercise 5.** ${}^{\bullet\bullet}$ Show that $\mathsf{DL}(k,\ell)$ is amenable iff $k=\ell$.

▷ **Exercise 6.** ${}^{\bullet\bullet\bullet}$ Consider the standard hexagonal lattice. Show that if you are given a bound $B<\infty$, and can group the hexagons into countries, each being a connected set of at most B hexagons, then it is not possible to have at least 7 neighbours for each country.

Figure 3: Trying to create at least 7 neighbours for each country.

▷ **Exercise 7.**

(a) ${}^{\bullet\bullet}$ Recall (or look it up in Durrett's book) that the reflection principle implies the following: if $\{X_k\}_{k\geq 0}$ is SRW on $\mathbb{Z}$, and $M_n=\max_{k\leq n} X_k$, then

$$2\,\mathbb{P}[\,X_n\geq t\,]\geq \mathbb{P}[\,M_n\geq t\,]\,.$$

Using this, prove that for SRW on the lamplighter group $\oplus_{\mathbb{Z}}\mathbb{Z}_2\rtimes\mathbb{Z}$, with the usual lazy generators (go left, go right, switch, do nothing), the return probability is at least $p_n(o,o)\geq \exp(-c\sqrt{n})$, for some absolute constant $c>0$. (Note that the subexponential decay corresponds to the graph being amenable.)

(b) ${}^{\bullet\bullet\bullet\bullet}$ Find a smarter version of the above strategy, giving $p_n(o,o)\geq\exp(-cn^{1/3})$, which is actually sharp.

The following three exercises together prove that the total variation mixing time (when the TV-distance goes below 1/4) of the 1/2-lazy random walk $X_0,X_1,\dots$ on the hypercube $\{0,1\}^k$ is $\sim\frac{1}{2}k\log k$. Then the fourth one proves that the uniform mixing time (when the L^∞-distance goes below $1/e$) is $\sim k\log k$.

▷ **Exercise 8.** ${}^{\bullet\bullet}$ Let Y_t be the number of missing coupons at time t in the coupon collector's problem with k coupons. Show that, for $\alpha\in(0,1)$ fixed,

$$\mathbb{E}\,Y_{\alpha\,k\log k}\sim k^{1-\alpha}\quad\text{and}\quad \mathbb{D}\,Y_{\alpha\,k\log k}=o(k^{1-\alpha}).$$

Using Markov's and Chebyshev's inequalities, deduce that $Y_{\alpha\,k\log k}/\sqrt{k}\to 0$ or ∞ in probability, for $\alpha>1/2$ and $<1/2$, respectively.

▷ **Exercise 9.** •• Let $\mathsf{N}(\mu, \sigma^2)$ denote the normal distribution. Show that, for any sequence $\sigma_k \to \sigma \in (0, \infty)$, we have that $d_{\mathrm{TV}}\big(\mathsf{N}(0, \sigma^2),\, \mathsf{N}(\mu_k, \sigma_k^2)\big) \to 0$ or 1, for $\mu_k \to 0$ and $\mu_k \to \infty$, respectively. Using this and the local version of the de Moivre–Laplace theorem, prove that

$$d_{\mathrm{TV}}\big(\mathsf{Binom}(k, 1/2),\, \mathsf{Binom}(k - k^\beta, 1/2) + k^\beta\big) \to \begin{cases} 0 & \text{if } \beta < 1/2\,, \\ 1 & \text{if } \beta > 1/2\,. \end{cases}$$

▷ **Exercise 10.**

(a) • For $X_0 = (0, 0, \dots, 0) \in \{0, 1\}^k$, let the distribution of X_t be μ_t. What is it, conditioned on $\|X_t\|_1 = \ell$?

(b) • What is the distribution of $\|Z\|_1$, where Z has distribution π, uniform on $\{0, 1\}^k$?

(c) •• Let Y_t be the number of coordinates that have not been rerandomized by time t in X_t. Compare the distribution of $k - \|X_t\|_1$, conditioned on $Y_t \geq y$, to $\mathsf{Binom}(k - y, 1/2) + y$. Deduce from the previous parts and the previous exercises that $d_{\mathrm{TV}}\big(\mu_{\alpha\, n \log n}, \pi\big) \to 0$ or 1, for $\alpha > 1/2$ and $< 1/2$, respectively.

▷ **Exercise 11.** •• Using Exercise 8, show that the uniform mixing time of the hypercube $\{0, 1\}^k$ is $\sim k \log k$.

▷ **Exercise 12.** •• Consider a reversible Markov chain P on a finite state space V with reversible distribution π and absolute spectral gap g_{abs}. This exercise explains why $T_{\mathrm{relax}} = 1/g_{\mathrm{abs}}$ is called the **relaxation time**.

Show that $g_{\mathrm{abs}} > 0$ implies that $\lim_{t\to\infty} P^t f(x) = \mathbb{E}_\pi f$ for all $x \in V$. Moreover,

$$\mathrm{Var}_\pi[P^t f] \leq (1 - g_{\mathrm{abs}})^{2t}\, \mathrm{Var}_\pi[f]\,,$$

with equality at the eigenfunction corresponding to the λ_i giving $g_{\mathrm{abs}} = 1 - |\lambda_i|$. Hence T_{relax} is the time needed to reduce the standard deviation of any function to $1/e$ of its original standard deviation.

▷ **Exercise 13.** This exercise explains why it is hard to construct large expanders. A *covering map* $\varphi : G' \longrightarrow G$ between graphs is a surjective graph homomorphism that is locally an isomorphism: denoting by $N_G(v)$ the subgraph induced by $v \in G$ and all its neighbours, we require that each connected component of the subgraph of G' induced by the full inverse image $\varphi^{-1}(N_G(v))$ be isomorphic to $N_G(v)$.

(a) • If $G' \longrightarrow G$ is a covering map of infinite graphs, then the spectral radii satisfy $\rho(G') \leq \rho(G)$, i.e., the larger graph is more non-amenable. In particular, if G is an infinite k-regular graph, then $\rho(G) \geq \rho(\mathbb{T}_k) = \frac{2\sqrt{k-1}}{k}$. (Hint: use the return probability definition of $\rho(G)$.)

(b) • If $G' \longrightarrow G$ is a covering map of finite graphs, then $\lambda_2(G') \geq \lambda_2(G)$, i.e., the larger graph is a worse expander. (Hint: eigenfunctions on G can be "lifted" to G'.)

▷ **Exercise 14.** • Give a sequence of d-regular transitive graphs $G_n = (V_n, E_n)$ with $|V_n| \to \infty$ that mix rapidly, $t_{\mathrm{mix}}^{\mathrm{TV}}(1/4) = O(\log |V_n|)$, but do not form an expander sequence. (You may accept here that transitive expanders do exist.)

▷ **Exercise 15.** Fix $d \in \mathbb{Z}_+$, take d independent uniformly random permutations $\pi_1, \dots, \pi_d$ on $[n]$, and consider the bipartite graph $V = [2n]$, $E = \{(v, n + \pi_i(v)) : 1 \leq i \leq d, 1 \leq v \leq n\}$.

(a) • Show that the number of multiple edges remains tight as $n \to \infty$.

(b) •• Show that, for every fixed r, the proportion of vertices whose r-neighbourhood is not a d-regular tree of depth r tends to 0 in probability.

(c) •• Show that there exists some $c = c_d > 0$, independent of n, such that the Cheeger constant of the graph is at least c with probability tending to 1.