Probability on Graphs and Groups — Third problem set

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The number of dots $^\bullet$ is the value of an exercise. Please hand in solutions, at least for 9 points, by June 20, Monday. Since this is very much the end of the semester, I'm trying to be somewhat generous with the exercise values; however, be careful: 8 means that I do not know the answer, while 5 means that I used to know the answer but don't remember it now.

▷ **Exercise 1.**

(a) $^{\bullet\bullet}$ Fix $o \in V(G)$ in a graph with maximal degree Δ. Prove that the number of connected sets $o \in S \subset V(G)$ of size n is at most $\Delta(\Delta-1)^{2n-3}$. (Hint: any S has a spanning tree, and one can "go around" a tree visiting each edge twice.) Conclude that $\mathbb{Z}^d$, $d \geq 2$, has an exponential bound on the number of minimal cutsets. In particular, $p_c(\mathbb{Z}^d) < 1$, although we already knew that from $\mathbb{Z}^2 \subseteq \mathbb{Z}^d$.

(b) $^{\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet}$ Let $\lambda(G) := \limsup_{n\to\infty} |\{S \subset V(G) : o \in S \text{ connected}, \ |S| = n\}|^{1/n}$ denote the exponential growth rate of the number of "lattice animals". We saw in part (a) that $\lambda(G) \leq (\Delta-1)^2$ for any graph of maximal degree Δ. What is the smallest possible upper bound here?

Note. Kesten's book *Percolation theory for mathematicians* (1982) has a beautiful argument proving $\lambda(G) \leq (\Delta-1)e$: for site percolation at $p = 1/(\Delta-1)$, write the probability that the cluster of o is finite using lattice animals and their outer vertex boundaries.

▷ **Exercise 2.** $^{\bullet\bullet\bullet}$ Consider a spherically symmetric tree T where each vertex on the n^{th} level T_n has $d_n \in \{k, k+1\}$ children, such that $\lim_{n\to\infty} |T_n|^{1/n} = k$, but $\sum_{n=0}^{\infty} k^n/|T_n| < \infty$. Using the second moment method, show that $p_c = 1/k$ and $\theta(p_c) > 0$.

The next exercise gives a non-trivial example of an invariant bond percolation on the 3-regular tree $\mathbb{T}_3$:

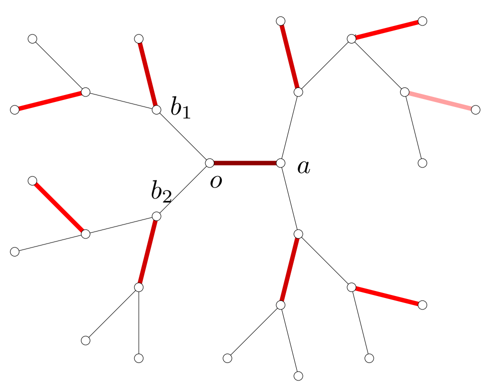

Figure 1: Constructing the unique invariant random perfect matching on $\mathbb{T}_3$.

▷ **Exercise 3.** $^{\bullet\bullet}$ We are going to define a random perfect matching ω of $\mathbb{T}_3$. Fix a root $o \in V(\mathbb{T}_3)$. Choose uniformly at random one of the three edges emanating from o to be in ω. Denote this chosen edge by (o, a), while the other two by (o, b_i), $i = 1, 2$. Now, for each b_i, choose one of the two edges emanating from b_i not in the direction of o uniformly at random to be in ω. The two edges emanating from a but not containing

o can of course not be in ω. Continue inductively for each vertex, for the pair of edges emanating from the vertex not in the direction of o. See Figure 1.

Prove that the resulting perfect matching has a distribution that is invariant under all the automorphisms $\mathrm{Aut}(\mathbb{T}_3)$, despite the fact that the construction itself used a specific vertex as the origin. Show that this is in fact the unique $\mathrm{Aut}(\mathbb{T}_3)$-invariant distribution on perfect matchings of $\mathbb{T}_3$.

A strengthening of ergodicity is **mixing**. For an invariant percolation $\mathbb{P}$ on an infinite transitive graph G, the definition is that for any two events A, B, and any $\epsilon > 0$, there is a finite set $K_\epsilon \subset \mathrm{Aut}(G)$ such that if $\gamma \notin K_\epsilon$, then $\big|\mathbb{P}[\, A \cap \gamma(B)\,] - \mathbb{P}[\, A\,]\mathbb{P}[\, B\,]\big| < \epsilon$. For general graphs, an appropriate definition is that, for any $k \geq 1$, if A and B_n are cylinder events such that each B_n depends on at most k variables (sites or bonds), and all the variables on which B_n depends are at distance at least n from the variables on which A depends, then $\lim_{n\to\infty} \big|\mathbb{P}[\, A \cap B_n\,] - \mathbb{P}[\, A\,]\mathbb{P}[\, B_n\,]\big| = 0$. For transitive graphs, the two definitions are easily seen to be equivalent. It is also easy to see that mixing implies ergodicity. Simple examples for ergodic but non-mixing percolations are periodic percolations. A non-periodic example comes from a key example of ergodic theory, irrational rotations:

▷ **Exercise 4.** ••• Consider the following site percolation on $\mathbb{Z}$: let $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ fixed, $U \sim \mathsf{Unif}[0, 1]$ random, and let $\omega_n := 1$ if $U + n\alpha \pmod 1$ is in $[0, 1/2)$, and $\omega_n := 0$ otherwise. Show that this is $\mathbb{Z}$-invariant, ergodic, but not mixing.

▷ **Exercise 5.**

- **(a)** •• Give an example of a unimodular transitive graph G such that there exist neighbours $x, y \in V(G)$ with no graph-automorphism interchanging them.
- **(b)** ••••• Can you give an example with a Cayley graph?

▷ **Exercise 6.**

- **(a)** •• In any invariant percolation process on a unimodular transitive graph G, show that almost surely the number of ends of each infinite cluster is 1 or 2 or continuum.
- **(b)** •• In any invariant percolation on any transitive amenable graph, show that there cannot be clusters with at least three ends. (In particular, any transitive graph with at least three ends is non-amenable.)
- **(c)** ••• Give an invariant percolation on a non-unimodular transitive graph that has infinite clusters with more than two but finitely many ends.

▷ **Exercise 7.** •• Give a non-amenable non-unimodular transitive graph for which, for any $\epsilon \in (0, 1)$, there exists an invariant percolation with edge marginals larger than $1 - \epsilon$, but no infinite clusters.

▷ **Exercise 8.** •• Show that any amenable group Γ has a Γ-invariant spanning $\mathbb{Z}$ on its vertices. (Not necessarily a subgraph of a given Cayley graph! Hint: use the invariant partitions into larger and larger finite pieces that we had in class.)