Stochastic models — First problem set

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Hand in 5 solutions out of the 14, by April 7. Beware: these problems are not very easy, so one or two days will not suffice. But I will be happy to give hints if you ask for help. Also note that the level of difficulty is not even: Exercise 6 is probably ten times harder than Exercise 5.

▷ **Exercise 1.** Let $X_0, X_1, X_2, \dots$ be SRW (simple random walk) on a locally finite graph, and let $D_n := \mathrm{dist}(X_n, X_0)$ be the graph distance from the starting point.

(a) Using the Central Limit Theorem, prove that $\mathbf{E}[\,D_n\,] \asymp \sqrt{n}$ on any $\mathbb{Z}^d$. (Be careful: convergence in distribution does not automatically imply convergence in L^1.)

(b) Comparing D_n on the d-regular tree $\mathbb{T}_d$ with a biased random walk on $\mathbb{Z}$, and using the exponential decay of the return probability $p_n(o,o)$ on $\mathbb{T}_d$, prove that $\lim_{n\to\infty} \mathbf{E}[\,D_n\,]/n = \frac{d-2}{d}$.

(c) For SRW on any *transitive* graph, show that the *speed* $\lim_{n\to\infty} \mathbf{E}[\,D_n\,]/n \in [0,1]$ exists.

▷ **Exercise 2.** Consider the two trees on Figure 1.

(a) On the left, a quasi-transitive tree, with degree 3 and degree 2 vertices alternating. Find the speed of SRW on it. You may use part (b) of the previous exercise.

(b) On the right, the so-called 3-1-tree, which has 2^n vertices on each level n, with the left 2^{n-1} vertices each having one child, the right 2^{n-1} vertices each having three children; the root has two children. Show that SRW on it is recurrent.

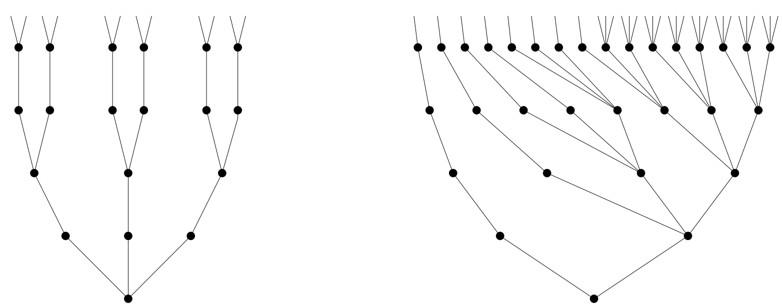

Figure 1: A quasi-transitive tree and the 3-1 tree.

▷ **Exercise 3.**

(a) Prove that for Green's function of simple random walk on a connected graph, $G(a,b|z) := \sum_{n\geq 0} p_n(a,b)\, z^n$, for any vertices x, y, a, b and any real $z > 0$,

$$G(x,y|z) < \infty \;\Leftrightarrow\; G(a,b|z) < \infty\,.$$

Therefore, by Pringsheim's theorem, we have that the radius of convergence is independent of x, y.

(b) Consider a reversible Markov chain on an infinite V, with constant reversible measure. Show that, for any $u, v \in V$,

$$\mathbf{P}_u[\,\tau_v < \infty\,] = \mathbf{P}_v[\,\tau_u < \infty\,]\,.$$

▷ **Exercise 4.** A simple version of the Tetris game (with no player): on the discrete cycle of length K, unit squares with sticky corners are falling from the sky, at places $[i, i+1]$ chosen uniformly at random ($i = 0, 1, \ldots, K-1$, mod K). Let R_t be the size of the roof after t squares have fallen: those squares of the current configuration that could have been the last to fall. Show that $\lim_{t\to\infty} \mathbf{E}R_t = K/3$.

Remark. If there are two types of squares, particles and antiparticles that annihilate each other when falling on exactly on top of each other, this process is a SRW on a group, and the size of the roof has to do with the speed of the SRW. Here, for $K \geq 4$, the expected limiting size of the roof is already less than $0.32893K$, but this is far from trivial. What's the situation for $K = 3$?

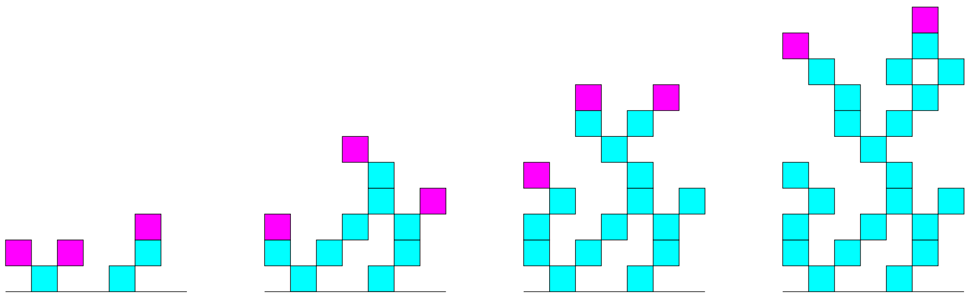

Figure 2: Sorry, this picture is on the segment, not on the cycle.

▷ **Exercise 5.** Recall (or look it up in Durrett's book) that the reflection principle implies the following: if $\{X_k\}_{k\geq 0}$ is SRW on $\mathbb{Z}$, and $M_n = \max_{k\leq n} X_k$, then

$$2\,\mathbf{P}[\,X_n \geq t\,] \geq \mathbf{P}[\,M_n \geq t\,]\,.$$

Using this, prove that for SRW on the lamplighter group $\oplus_{\mathbb{Z}}\mathbb{Z}_2 \rtimes \mathbb{Z}$, with the usual lazy generators (go left, go right, switch, do nothing), the return probability is at least $p_n(o,o) \geq \exp(-c\sqrt{n})$, for some absolute constant $c > 0$. (Note that the subexponential decay corresponds to the graph being amenable.)

Remark. You may try to find a smarter version of the above strategy, giving $p_n(o,o) \geq \exp(-cn^{1/3})$, which is actually sharp.

▷ **Exercise 6.** Consider the standard hexagonal lattice. Show that if you are given a bound $B < \infty$, and can group the hexagons into countries, each being a connected set of at most B hexagons, then it is not possible to have at least 7 neighbours for each country.

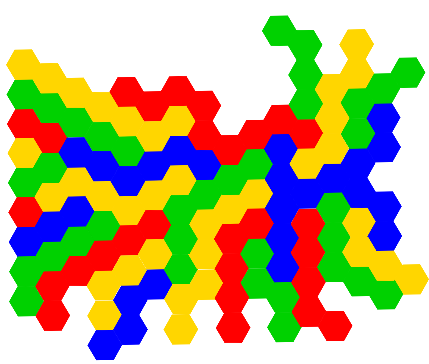

Figure 3: Trying to create at least 7 neighbours for each country.

▷ **Exercise 7.** This exercise explains why it is hard to construct large expanders. A *covering map* $\varphi : G' \longrightarrow G$ between graphs is a surjective graph homomorphism that is locally an isomorphism: denoting by $N_G(v)$ the subgraph induced by $v \in G$ and all its neighbours, we require that each connected component of the subgraph of G' induced by the full inverse image $\varphi^{-1}(N_G(v))$ be isomorphic to $N_G(v)$.

(a) If $G' \longrightarrow G$ is a covering map of infinite graphs, then the spectral radii satisfy $\rho(G') \le \rho(G)$, i.e., the larger graph is more non-amenable. In particular, if G is an infinite k-regular graph, then $\rho(G) \ge \rho(\mathbb{T}_k) = \frac{2\sqrt{k-1}}{k}$. (Hint: use the return probability definition of $\rho(G)$.)

(b) If $G' \longrightarrow G$ is a covering map of finite graphs, then $\lambda_2(G') \ge \lambda_2(G)$, i.e., the larger graph is a worse expander. (Hint: eigenfunctions on G can be "lifted" to G'.

▷ **Exercise 8.** Consider a reversible Markov chain P on a finite state space V with stationary distribution π and absolute spectral gap g_{abs}. This exercise explains why $\tau_{\mathrm{relax}} = 1/g_{\mathrm{abs}}$ is called the relaxation time.

(a) For $f : V \longrightarrow \mathbb{R}$, let $\mathrm{Var}_\pi[f] := \mathbf{E}_\pi[f^2] - (\mathbf{E}_\pi f)^2 = \sum_x f(x)^2\pi(x) - \big(\sum_x f(x)\pi(x)\big)^2$. Show that $g_{\mathrm{abs}} > 0$ implies that $\lim_{t\to\infty} P^t f(x) = \mathbf{E}_\pi f$ for all $x \in V$. Moreover,

$$\mathrm{Var}_\pi[P^t f] \le (1 - g_{\mathrm{abs}})^{2t}\, \mathrm{Var}_\pi[f]\,,$$

with equality at the eigenfunction corresponding to the λ_i giving $g_{\mathrm{abs}} = 1 - |\lambda_i|$. Hence τ_{relax} is the time needed to reduce the standard deviation of any function to $1/e$ of its original standard deviation.

(b) Using part (a), prove that there is a universal constant $C < \infty$ such that $\tau_{\mathrm{relax}} < C\,\tau_{\mathrm{mix}}^{\mathrm{TV}}$.

▷ **Exercise 9.** This exercise proves that the total variation mixing time of the 1/2-lazy random walk $X_0, X_1, \dots$ on the hypercube $\{0,1\}^n$ is $(1/2 + o(1))\, n \log n$.

(a) Let Y_t be the number of missing coupons at time t in the coupon collector's problem. Show that $\mathbf{E}\, Y_{\alpha\, n \log n} \sim n^{1-\alpha}$ and $\mathbb{D}\, Y_{\alpha\, n\log n} = o(n^{1-\alpha})$. Using Markov's and Chebyshev's inequalities, deduce that $Y_{\alpha\, n \log n}/\sqrt{n} \to 0$ or ∞ in probability, for $\alpha > 1/2$ and $< 1/2$, respectively.

(b) Show that $d_{\mathrm{TV}}\big(\mathsf{N}(0,1), \mathsf{N}(x,1)\big) \to 0$ or 1, for $x \to 0$ and $x \to \infty$, respectively, where $\mathsf{N}(\mu, \sigma^2)$ is the normal distribution. Using this and the local version of the de Moivre–Laplace theorem, prove that $d_{\mathrm{TV}}\big(\mathsf{Binom}(n, 1/2), \mathsf{Binom}(n - n^\beta, 1/2) + n^\beta\big) \to 0$ for any fixed $\beta < 1/2$, while $\to 1$ for $\beta > 1/2$.

(c) For $X_0 = (0, 0, \dots, 0) \in \{0,1\}^n$, let the distribution of X_t be μ_t. What is it, conditioned on $\|X_t\|_1 = k$? And what is the distribution of $\|Z\|_1$, where Z has distribution π, uniform on $\{0,1\}^n$?

(d) Deduce from the previous parts that $d_{\mathrm{TV}}\big(\mu_{\alpha\, n\log n}, \pi\big) \to 0$ or 1, for $\alpha > 1/2$ and $< 1/2$, respectively.

▷ **Exercise 10.** Let T be the Galton-Watson tree with offspring distribution $\xi \sim \mathsf{Geom}(1/2)$. Draw the tree into the plane with root ρ, add an extra vertex ρ' and an edge (ρ, ρ'), and walk around the tree, starting from ρ', going through each "corner" of the tree once, through each edge twice (once on each side). At each corner visited, consider the graph distance from ρ': let this be process be $\{X_t\}_{t=0}^{2n}$, which is positive everywhere except at $t = 0, 2n$, where n is the number of vertices of the original tree T.

Figure 4: The contour walk around a tree.

(a) Using the memoryless property of $\mathsf{Geom}(1/2)$, show that $\{X_t\}$ is SRW on $\mathbb{Z}$.

(b) Using martingale techniques, show that $\mathbf{P}[T \text{ has height} \ge n] = 1/n$.

(c) Show that, conditioning T to have height at least n, with high probability the height will be around n and the total volume will be around n^2, where "around" means "up to constant factors".

Let G_n be a sequence of finite graphs with degrees at most d. Pick a uniform random root ρ_n from $V(G_n)$, and take the ball $B_{G_n,\rho_n}(r)$ around it in the graph metric, with some fixed radius $r \in \mathbb{Z}_+$. This way we get a distribution $\mu_{n,r}$ on finite rooted graphs with degrees at most d. We say that the sequence $\{G_n\}$ converges in the **Benjamini-Schramm sense** (also called **local weak convergence**) to a random rooted graph (G, ρ), if, for every r, the distributions $\mu_{n,r}$ converge weakly as $n \to \infty$ to the distribution of $B_{G,\rho}(r)$. The simplest case is that the limit is a transitive infinite graph G: the measures $\mu_{n,r}$ converge to the Dirac measure on a single graph, the r-ball of G.

▷ **Exercise 11.**

(a) Prove that the cubes $\{1, \dots, n\}^d$ converge to $\mathbb{Z}^d$ in the local weak sense.

(b) Find a random rooted graph (G, ρ) that is a local weak limit of the balls $G_n = B_{\mathbb{T}_d,o}(n)$ in the d-regular tree $\mathbb{T}_d$. (Note that the limit will not be $\mathbb{T}_d$, or any other transitive graph, since ρ_n is a leaf of G_n with a uniformly positive probability, which will be inherited to ρ.)

More generally:

▷ **Exercise 12.** Show that a transitive graph G has a sequence G_n of *subgraphs* converging to it in the local weak sense iff it is amenable.

▷ **Exercise 13.** Show that the random d-regular bipartite graphs from class converge to the d-regular tree $\mathbb{T}_d$ in the local weak sense. (Here the randomness for the measure $\mu_{n,r}$ comes from two sources: we take a random root ρ_n in the random graph G_n.)

The phenomenon is the same as in the previous exercise, but the computation is a bit simpler:

▷ **Exercise 14.** Show that for any $\lambda \in \mathbb{R}_+$, the local weak limit of the Erdős-Rényi random graphs $G(n, \lambda/n)$ is the $\mathsf{PGW}(\lambda)$ tree: the Galton-Watson tree with $\mathsf{Poisson}(\lambda)$ offspring distribution, rooted as normally.