# Stochastic Models — First HW problem set

Gábor Pete
http://www.math.bme.hu/~gabor

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The number of dots $^\bullet$ is the value of an exercise. **Hand in solutions for 12 points by April 27 Thursday in class.** If you have seriously tried to solve some problem, but got stuck, I will be happy to help. Also, if your final solution to a problem has some mistake but has some potential to work, then I will give it back and you can try and correct the mistake.

▷ **Exercise 1.** Let $G(V,E)$ be a connected infinite graph.

**(a)**$^\bullet$ Show that if simple random walk started at some $o \in V$ visits $o$ infinitely often almost surely, then the walk started at any $x \in V$ visits any given $y \in V$ infinitely often, almost surely. Consequently, recurrence is a property solely of the graph.

**(b)**$^\bullet$ If the graph is transient, then the walk visits any given finite set only finitely many times.

▷ **Exercise 2.**$^{\bullet\bullet}$ Give an example of an iid random walk on $\mathbb{Z}$ with symmetric jump distribution that is transient. (Hint: the explicit example you will find on the internet is missing the main step in the calculation, so it's no good. But you don't need to give the jump probabilities explicitly. Instead, use that simple random walk on $\mathbb{Z}^2$ is recurrent, on $\mathbb{Z}^3$ is transient, and you don't need any further calculations.)

▷ **Exercise 3.**$^\bullet$ Show that if $\{M_i\}_{i=0}^\infty$ is a martingale, then the differences $X_i = M_i - M_{i-1}$ satisfy the uncorrelatedness condition $\mathbf{E}\big[\, X_{i_1} \cdots X_{i_k} \,\big] = 0$, for any $k \in \mathbb{Z}_+$ and $i_1 < i_2 < \cdots < i_k$.

▷ **Exercise 4.**$^{\bullet\bullet}$ Prove that for Green's function of simple random walk on a connected graph, $G(a,b|z) := \sum_{n\geq 0} p_n(a,b)\, z^n$, for any vertices $x,y,a,b$ and any real $z>0$,

$$G(x,y|z) < \infty \;\Leftrightarrow\; G(a,b|z) < \infty\,.$$

Deduce, using Pringsheim's theorem, that the radius of convergence is independent of $x,y$.

▷ **Exercise 5.** Let $D_n := \mathrm{dist}(X_n, X_0)$ be the distance of SRW from the starting point on an infinite graph.

**(a)**$^{\bullet\bullet}$ Using the Central Limit Theorem, prove that $\mathbf{E}[\, D_n \,] \asymp \sqrt{n}$ on any $\mathbb{Z}^d$. (Note that convergence in distribution does *not* automatically imply the convergence of expectations.)

**(b)**$^{\bullet\bullet}$ Using the transience of the SRW on $\mathbb{T}_k$, $k\geq 3$, which we have proved in class, show that $D_n/n \to \frac{k-2}{k}$ almost surely, and $\mathbf{E}[\, D_n \,] \sim \frac{k-2}{k} n$, as $n\to\infty$.

▷ **Exercise 6.**$^{\bullet\bullet}$ Let $\mathbb{T}_{k,\ell}$ be the tree where, if $v_n \in \mathbb{T}_{k,\ell}$ is a vertex at distance $n$ from the root, then

$$\deg v_n = \begin{cases} k & \text{if } n \text{ is even} \\ \ell & \text{if } n \text{ is odd}\,. \end{cases}$$

Show the almost sure limiting speed $\lim_{n\to\infty} d(X_0, X_n)/n$ exists, and compute its value.

▷ **Exercise 7.**$^{\bullet\bullet\bullet}$ Compute the spectral radius $\rho(\mathbb{T}_{k,\ell})$ for the previous tree. Please check that your formula for $k=\ell$ gives back $\rho(\mathbb{T}_k) = 2\sqrt{k-1}/k$, or I won't read your solution.

▷ **Exercise 8.** Take a connected infinite transitive graph $G$ with bounded degree. Let $\Gamma_n$ be the number of self-avoiding paths (no repeated vertices) of length $n$ starting from a fixed vertex.

**(a)**$^\bullet$ Show that $\gamma(G) := \lim_{n\to\infty} \Gamma_n^{1/n}$ exists and is finite. This is called the connective constant of the graph $G$.

**(b)**$^\bullet$ Show that $2 \le \gamma(\mathbb{Z}^2) \le 3$.

▷ **Exercise 9.**$^\bullet$ In First Passage Percolation on a graph $G(V,E)$, we assign iid nonnegative random weights $\omega_e$ to the edges $e \in E$, then study the resulting random metric $\mathrm{dist}_\omega(\cdot,\cdot)$ on $V \times V$, where the length of each edge is not 1, but its weight. Let the graph be $\mathbb{Z}^2$, and let the weight distribution be $\mathbf{P}[\,\omega_e = a\,] = 1 - \mathbf{P}[\,\omega_e = b\,] = p$, with some fixed $0 < a < b < \infty$ and $p \in (0,1)$. Let $L_n := \mathbf{E}\big[\,\mathrm{dist}_\omega\big((0,0),(n,n)\big)\,\big]$. Show that $\lim_n L_n/n$ exists and is positive and finite.

Recall that a bounded degree infinite graph satisfies the isoperimetric inequality $IP_d$ if there exists some $c > 0$ such that $|\partial S| > c|S|^{\frac{d-1}{d}}$ for every finite $S \subset V(G)$, where $\partial S$ can stand for edge boundary, or inner or outer vertex boundary, as you wish. In particular, $IP_\infty$ means non-amenable.

▷ **Exercise 10.**$^\bullet$ Show that if $S_1$ and $S_2$ are two finite generating sets of the same group $\Gamma$, and $\mathrm{Cayley}(\Gamma, S_1)$ satisfies some $IP_d$, then so does $\mathrm{Cayley}(\Gamma, S_2)$.

▷ **Exercise 11.**$^{\bullet\bullet}$ Show that a bounded degree tree without leaves is amenable iff there is no bound on the length of "hanging chains", i.e., chains of neighbouring vertices with degree 2. (Consequently, for trees, $IP_{1+\epsilon}$ implies $IP_\infty$.)

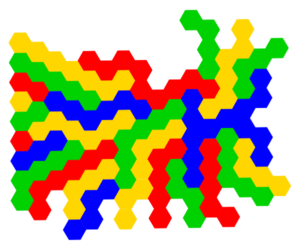

Figure 1: Trying to create at least 7 neighbours for each country.

▷ **Exercise 12.**$^{\bullet\bullet\bullet\bullet}$ Consider the standard hexagonal lattice. Show that if you are given a bound $B < \infty$, and can group the hexagons into countries, each being a connected set of at most $B$ hexagons, then it is not possible to have at least 7 neighbours for each country.

The **3-dimensional discrete Heisenberg group** is the matrix group

$$H_3(\mathbb{Z}) = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \;:\; x,y,z \in \mathbb{Z} \right\}.$$

If we denote by $X, Y, Z$ the matrices given by the three permutations of the entries $1,0,0$ for $x,y,z$, then $H_3(\mathbb{Z})$ is given by the presentation $\big\langle X,Y,Z \,\big|\, [X,Z]=1, [Y,Z]=1, [X,Y]=Z \big\rangle$, where $[a,b]=aba^{-1}b^{-1}$.

▷ **Exercise 13.** We say that a bounded degree graph $G(V,E)$ has $d$-dimensional volume growth if there exist $0 < c < C < \infty$ such that $cr^d < |B_r(o)| < Cr^d$ for any $o \in V$ and every large enough $r > r^*(o)$.

**(a)**$^\bullet$ Show that if a group has a finitely generated Cayley graph with $d$-dimensional volume growth, then all its Cayley graphs have $d$-dimensional volume growth.

**(b)**$^{\bullet\bullet}$ Show that the discrete Heisenberg group has 4-dimensional volume growth.

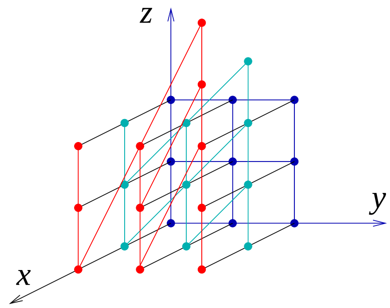


Figure 2: The Cayley graph of the Heisenberg group with generators $X, Y, Z$.

▷ **Exercise 14.** Recall (or look it up in Durrett's book) that the reflection principle implies the following: if $\{X_k\}_{k\geq 0}$ is SRW on $\mathbb{Z}$, and $M_n = \max_{k\leq n} X_k$, then

$$2\,\mathbf{P}[\,X_n \geq t\,] \geq \mathbf{P}[\,M_n \geq t\,]\,.$$

Consider now SRW on the lamplighter group $\mathbb{Z}_2 \wr \mathbb{Z}$, with the generators Left, Right, Switch, Nothing, each with probability 1/4.

**(a)**$^{\bullet\bullet}$ Prove that the return probability is at least $p_n(o,o) \geq \exp(-c\sqrt{n})$, for some absolute constant $c > 0$. (Note that the subexponential decay corresponds to the graph being amenable.)

**(b)**$^{\bullet\bullet\bullet\bullet}$ Find a smarter version of this strategy and prove $p_n(o,o) \geq \exp(-cn^{1/3})$. This is actually the sharp exponent (but that you don't have to give any upper bound).

▷ **Exercise 15.**$^{\bullet\bullet\bullet}$ A simple version of the Tetris game (with no player): on the discrete cycle of length $K$, unit squares with sticky corners are falling from the sky, at places $[i, i+1]$ chosen uniformly at random ($i = 0, 1, \ldots, K-1$, mod $K$). Let $R_t$ be the size of the roof after $t$ squares have fallen: those squares of the current configuration that could have been the last to fall. Show that $\lim_{t\to\infty} \mathbf{E}R_t = K/3$.

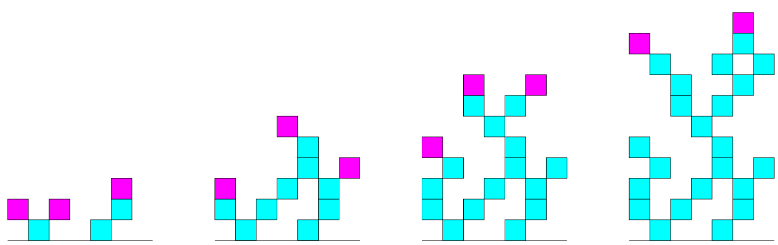

Figure 3: Sorry, this picture is on the segment, not on the cycle.

**Remark.**$^{\bullet}$ If there are two types of squares, particles and antiparticles that annihilate each other when falling on exactly on top of each other, this process is a SRW on a group, and the size of the roof has to do with the speed of the SRW. Here, for $K \geq 4$, the expected limiting size of the roof is already less than $0.32893K$, but this is far from trivial. What's the situation for $K = 3$?