Stochastics

Problem sheet 3 - Generating functions, branching processes

Fall 2022

1. A nonnegative discrete random variable has generating function
$$G(z) = \frac{3}{8} + \frac{3}{8}z + \frac{1}{8}z^2 + \frac{1}{8}z^3.$$
Determine the distribution of X (that is, the $\mathbb{P}(X = k)$ probabilities for $k = 0, 1, 2, \dots$). Calculate its mean and variance as well.

2. Alice sends a letter to Bob. Postal service is not very reliable; each day, the postman will take the letter to the logistics center with probability 1/3 (regardless of the past). Once the letter is in the logistics center, each day it is processed with probability 1/5 (regardless of the past). Once it is processed, shipping it takes 1 day. (So at best, the total delivery time is 1 day.) Let X denote the total delivery time in days. Calculate the generating function and the mean of X.

3. An exam has two parts, A and B. Part B of the exam may be taken only by students who pass part A. Each student passes part A with probability 0.6, independent of the others. Each student who passed part A then passes part B with probability 0.5, independent of the others. 100 students take this test. Let X denote the number of students who pass part A, and Y denote the number of students who pass part B. What is the distribution of X? Calculate G_X, the generating function of X, then derive G_Y, the generating function of Y using G_X. Can we tell the distribution of Y from G_Y?

4. Let $X_1, X_2, \dots$ be iid random variables and N a discrete random variable, independent from the X's, and let $Y = \max(X_1, \dots, X_N)$. Express the cumulative distribution function of Y using the common cumulative distribution function of the X's and the generating function of N.

5. On average, 2 cars per minute pass on a low traffic road. On average, 20% of the cars are red. What is the distribution of the number of red cars passing in a 1 minute interval? (Hint: let X denote the total number of cars passing in the 1 minute interval, and let Y denote the number of red cars passing in the same 1 minute interval. Write Y as a sum with a random number of terms.)

6. Between 8 and 9 am the number X of requests arriving to the router has Poisson distribution with parameter λ. Each request may come independently of each other from places A and B with probability p and $(1-p)$ respectively. What is the distribution of the number of requests coming from place A?

7. Let $X_1, X_2, \dots$ be i.i.d. $\mathbb{N}$ valued random variables. Furthermore, let N be an $\mathbb{N}$ valued random variable which is independent of the Xs. Let Y be equal to $\sum_{i=1}^{N} X_i$. Show that
   - $E(Y) = E(N)E(X_1)$,
   - the variance $D^2(Y)$ of Y is equal to $D^2(N)(E(X_1))^2 + E(N)D^2(X_1)$.

8. We roll two fair dice a blue and a red. We first roll the red die then we roll the blue die as many times as the outcome of the red die. Let Y denote the outcome of the red die and denote by X the sum of the outcomes on the blue die.

   (a) Find $E(X)$ and $Var(X)$.

   (b) What is the sign of $Cov(X, Y)$?

9.
   - We keep rolling a fair die until we first roll a 6. Let X denote the number of rolls (including the roll of 6). Find the generating function of X using recursion coming from conditioning on the result of the first roll.
   - We keep rolling a fair die until we first roll two 6's in a row. Let Y denote the number of rolls (including the two rolls of 6 in the end). Find the generating function of Y using recursion coming from conditioning on the result of the next roll after the first roll of 6.

10. We keep rolling a fair die until we first roll a 6. Let X denote the sum of the numbers rolled before (and not including) that 6. Calculate

    - the generating function of X,
    - the expectation of X,
    - the variance of X.

    (Warning: What is the conditional distribution of a number rolled under the condition that it is not a 6?)

11. * Design two fair 6-sided dice X_1 and X_2 such that $X_1 + X_2$ has the same distribution as the sum of 2 rolls with a regular fair 6-sided die, both are fair, all sides have nonnegative integer numbers, but X_1 and X_2 are different.

12. In a very large community, initially one person is infected with an infectious disease. Before he gets cured, he infects X_1 other people, where $X_1 \sim \text{PGEO}(p)$. After he is cured, he is not infectious anymore. Each further infected person infects more people with the same distribution and independently from the others. Model this scenario with a branching process, and answer the following questions for both $p = 0.4$ and $p = 0.6$.

    (a) What is the mean of X_1?

    (b) What is the probability, that apart from the initial person, nobody else infects others (that is, generation 2 is empty)?

    (c) Let X_3 denote the number of people in generation 3. Calculate $\mathbb{E}X_3$.

    (d) Calculate the probability that none of the generations are empty. (This corresponds to an epidemic.)

    (e) Let N denote the total number of people infected. Calculate $\mathbb{E}N$.

13. The teacher gives Jack problems to solve. Jack can solve each problem correctly with probability 1/2, independently from the others. For each incorrect solution, the teacher gives him two more problems to solve. Jack starts with a single problem to solve. What is the probability that Jack will eventually finish solving the problems? What is the expected number of problems he has to solve before he finishes?

14. Let the offspring distribution of a branching process have probability generating function $G(z)$. Calculate the following conditional probabilities using G.

    (a) The conditional probability of extinction, assuming that the first generation has size k.

    (b) The conditional probability of extinction, assuming that the first generation is not empty.

15. Let $\Theta(p)$ denote the probability of extinction for a branching process with offspring distribution PGEO(p). Calculate the function $\Theta(p)$ for $0 < p < 1$.

16. Anna makes up a joke. She tells to a random number of people. She doesn't tell it to anybody with probability 1/2, she tells it to 1 person with probability 1/4 and to 2 people with probability 1/4. Each person that hears the joke will tell it to a random number of people with the same distribution, independently from the others. Let generation 1 denote the people who heard the joke from Anna, generation 2 the people who heard it from someone in generation 1 and so on. X_k denotes the number of people in generation k.

    (a) What is the probability that $X_2 = 0$?

    (b) What is the expectation of X_3?

    (c) What is the probability that the joke stops spreading eventually?

    (d) What is the expected number of people who hear the joke?

17. A nuclear reactor contains a large number of nuclei that split when hit by a neutron. The fate of a neutron in the reactor may be the following:

    - it is absorbed in a non-splitting nucleus with probability p, or
    - it splits a nucleus; the original neutron is absorbed, but 1, 2 or 3 new neutrons are released with equal probability $\frac{1-p}{3}$.

    The fate of each neutron is independent from all other neutrons and also from the past. The value of p depends on the size and shape of the reactor and also on the use of regulatory devices.

    We fire a single neutron in the reactor. Let generation 1 be the neutrons released by the original single neutron (if there are any), generation 2 are the neutrons released by the neutrons in generation 1 and so on. The number of neutrons in generation k is denoted by X_k. N denotes the total number of neutrons in the reactor throughout the process.

    Answer the following questions for both $p = 1/4$ and $p = 5/8$.

    (a) Calculate the generating function of X_1 and X_2.

    (b) $\mathbb{E}X_1 =$? $\mathbb{E}X_{10} =$?

    (c) $\mathbb{P}(X_4 = 0) =$?

    (d) $\mathbb{P}(N < \infty) =$? $\mathbb{E}N =$?

18. In medieval times, a nobleman has a number of sons with offspring distribution PGEO(0.4) who carry on his family name. Then each of his male children will have a random number of sons (who carry on the family name) with the same distribution, independent from everybody else, and so on.

(a) Model the scenario with a branching process. What is the expectation of the offspring distribution? Is the process subcritical, critical or supercritical?

(b) What is the expected number of the male grandchildren of the original nobleman?

(c) What is the probability that the nobleman's family name dies out eventually?

19. * Let $\mu > 1$, and consider the distribution of a branching process with offspring distribution $\mathrm{POI}(\mu)$, *conditioned on extinction*. Prove that this has the same distribution as a branching process with offspring distribution $\mathrm{POI}(\lambda)$, where λ is the solution of $\lambda e^{-\lambda} = \mu e^{-\mu}$ for which $\lambda < 1$. This is known as the duality of the Poisson branching process. (Mind that the outcomes of the branching process are finite or infinite trees, and its distribution means the probability measure with which the branching process gives given trees as realizations.)

HW4 (Deadline: 19 Oct. 12:15) A chain letter e-mail asks its reader to forward it to 12 other people. 92% of the people delete the letter without forwarding it; however, 8% do forward it to 12 other people.

(a) Model the scenario with a branching process. What is the expectation of the offspring distribution? Is the process subcritical, critical or supercritical?

(b) What is the probability that the chain letter will stop spreading eventually?

(c) Assuming that the creator of the chain letter sent it to 12 other people, what is the expected number of e-mails sent throughout the entire „lifetime" of the chain letter?