

IMPROVED BOUNDS ON THE SUPREMUM OF AUTOCONVOLUTIONS

MÁTÉ MATOLCSI, CARLOS VINUESA

ABSTRACT. We give a slight improvement of the best known lower bound for the supremum of autoconvolutions of nonnegative functions supported in a compact interval. Also, by means of explicit examples we disprove a long standing natural conjecture of Schinzel and Schmidt concerning the extremal function for such autoconvolutions.

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1. INTRODUCTION

Consider the set $\mathcal{F}$ of all nonnegative real functions f with integral 1, supported on the interval $[-\frac{1}{4}, \frac{1}{4}]$. What is the minimal possible value for the supremum of the autoconvolution $f * f$? It has repeatedly been conjectured $\pi/2$ ([9], [8]), which was the starting point and motivation for the present work. The problem can also be stated in a probabilistic language. We can regard f as the density function of two identically distributed random variables, X and Y . Then, the density function of $X + Y$ is given by the autoconvolution $f * f$, and we are asking for the infimum of the supremum of the density function of the sum $X + Y$.

Define the autoconvolution of f as

$$f * f(x) = \int f(t)f(x - t) dt.$$

We are thus interested in

$$S = \inf_{f \in \mathcal{F}} \|f * f\|_{\infty}$$

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where the infimum is taken over all functions f satisfying the above restrictions.

This question (or equivalent formulations of it) has been studied in several papers recently [9, 7, 10, 8], and is motivated by its discrete analogue, the study of the maximal possible cardinality of g -Sidon sets (or $B_2[g]$ sets) in $\{1, \dots, n\}$. The connection between $B_2[g]$ sets and autoconvolutions is described (besides several additional results) in [7, 3, 2].

To be more explicit, given a set of integers, A , we define the representation function, $r(x) = \#\{(a_1, a_2) : a_1 + a_2 = x, a_i \in A\}$, and we say that A is a g -Sidon set if $r(x) \leq g$ for every integer x .

In 1932, Simon Sidon asked Paul Erdős about the maximal cardinality of a g -Sidon set in $\{1, \dots, n\}$. Since then, this problem has been one of the main topics in Combinatorial Number Theory.

We can define $\beta_g(n) = \max\{|A| : A \subseteq \{1, \dots, n\}, A \text{ is a } g\text{-Sidon set}\}$. Although it is known that $\beta_2(n) \sim \sqrt{n}$ and $\beta_3(n) \sim \sqrt{n}$, for $g \geq 4$ it has not even been proved that $\lim_{n \rightarrow \infty} \beta_g(n)/\sqrt{n}$ exists. This is the reason to define $\bar{\beta}_g = \limsup_{n \rightarrow \infty} \beta_g(n)/\sqrt{n}$ and $\underline{\beta}_g = \liminf_{n \rightarrow \infty} \beta_g(n)/\sqrt{n}$.

It is not difficult to prove that $\bar{\beta}_g \leq \sqrt{2g}$ and $\underline{\beta}_g \geq \sqrt{g/2}$. The problem of narrowing this gap has attracted the attention of many mathematicians in the last years. The successive upper bounds have been:

$$\begin{aligned} \frac{\bar{\beta}_g}{\sqrt{g}} &\leq \sqrt{2} = 1.4142\dots \text{ (trivial)} \\ &\leq 1.3180\dots \text{ (J. Cilleruelo - I. Z. Ruzsa - C. Trujillo, [1])} \\ &\leq 1.3039\dots \text{ (B. Green, [4])} \\ &\leq 1.3003\dots \text{ (G. Martin - K. O'Bryant, [7])} \\ &\leq 1.2649\dots \text{ (G. Yu, [10])} \\ &\leq 1.2588\dots \text{ (G. Martin - K. O'Bryant, [8])} \\ &\leq \sqrt{2/S} \text{ (A. Schinzel - W. M. Schmidt, [9])}. \end{aligned}$$

For large g , the successive lower bounds have been:

$$\begin{aligned} \lim_{g \rightarrow \infty} \frac{\beta_g}{\sqrt{g}} &\geq 1/\sqrt{2} = 0.7071\dots \text{ (M. Kolountzakis, [5])} \\ &\geq 0.75 \text{ (J. Cilleruelo - I. Z. Ruzsa - C. Trujillo, [1])} \\ &\geq 0.7933\dots \text{ (G. Martin - K. O'Bryant, [6])} \\ &\geq 1/\sqrt{S} \text{ (J. Cilleruelo - C. Vinuesa, [3])} \\ &\geq \sqrt{2/S} \text{ (J. Cilleruelo - I. Z. Ruzsa - C. Vinuesa, [2])}. \end{aligned}$$

The combination of the last inequalities is the main result of [2], namely

$$\lim_{g \rightarrow \infty} \frac{\beta_g}{\sqrt{g}} = \lim_{g \rightarrow \infty} \frac{\bar{\beta}_g}{\sqrt{g}} = \sqrt{2/S},$$

which shows explicitly the relation between g -Sidon sets and the constant S .

This short note gives two contributions to the subject. On the one hand, in Section 3 we improve the best known lower bound on S . This is achieved by following the ideas of Yu [10], and Martin & O'Bryant [8], and improving them in two minor aspects. On the other hand, maybe more interestingly, Section 4 provides counterexamples to a long-standing natural conjecture of Schinzel and Schmidt [9] concerning the extremal function for such autoconvolutions. In some sense these examples open up the subject considerably: at this point we do not have any natural conjectures for the exact value of S or any extremal functions where this value could be attained. Upon numerical evidence we are inclined to believe that $S \approx 1.5$, unless there exists some hidden “magical” number theoretical construction yielding a much smaller value (the possibility of which is by no means excluded).

In short, we will prove

$$(1) \quad 1.2748 \leq S \leq 1.5098$$

which improves the best lower and upper bounds that were known for S . It also implies the improved bounds

$$(2) \quad 1.1509... \leq \lim_{g \rightarrow \infty} \frac{\beta_g}{\sqrt{g}} = \lim_{g \rightarrow \infty} \frac{\bar{\beta}_g}{\sqrt{g}} \leq 1.2525....$$

2. NOTATION

Throughout the paper we will use the following notation (mostly borrowed from [8]).

Let $\mathcal{F}$ denote the set of nonnegative real functions f supported in $[-1/4, 1/4]$ such that $\int f(x) dx = 1$. We define the autoconvolution of f , $f * f(x) = \int f(t)f(x-t) dt$ and its autocorrelation, $f \circ f(x) = \int f(t)f(x+t) dt$. We are interested in $S = \inf_{f \in \mathcal{F}} \|f * f\|_{\infty}$. We remark here that the value of S does not change if one considers nonnegative step functions in $\mathcal{F}$ only. This is proved in Theorem 1 in [9]. Therefore the reader may assume that f is square integrable whenever this is needed.

We will need a parameter $0 < \delta \leq 1/4$ and use the notation $u = 1/2 + \delta$, and $\tilde{g}(\xi) = \frac{1}{u} \int_{\mathbb{R}} g(x) e^{-2\pi i x \xi / u} dx$ for any function g . We will also use Fourier coefficients of period 1, i.e. $\hat{g}(\xi) = \int_{\mathbb{R}} g(x) e^{-2\pi i x \xi} dx$ for any function g .

We will need a nonnegative kernel function K supported in $[-\delta, \delta]$ with $\int K = 1$. We will also need that $\tilde{K}(j) \geq 0$ for every integer j . We are quite convinced that the choice of K in [8] is optimal, and we will not change it (see equation (7) below).

3. AN IMPROVED LOWER BOUND

We will follow the steps of [8] (which, in turn, is based on [10]). We include here all the ingredients for convenience (the proofs can be found in [8]).

Lemma 3.1. [Lemmas 3.1, 3.2, 3.3, 3.4 in [8]] *With the notation f, K, δ, u as described above, we have*

$$(3) \quad \int (f * f(x)) K(x) dx \leq \|f * f\|_{\infty}.$$

$$(4) \quad \int (f \circ f(x)) K(x) dx \leq 1 + \sqrt{\|f * f\|_{\infty} - 1} \sqrt{\|K\|_2^2 - 1}.$$

$$(5) \quad \int (f * f(x) + f \circ f(x)) K(x) dx = \frac{2}{u} + 2u^2 \sum_{j \neq 0} (\Re \tilde{f}(j))^2 \tilde{K}(j).$$

Let G be an even, real-valued, u -periodic function that takes positive values on $[-1/4, 1/4]$, and satisfies $\tilde{G}(0) = 0$. Then

$$(6) \quad u^2 \sum_{j \neq 0} (\Re \tilde{f}(j))^2 \tilde{K}(j) \geq \left(\min_{0 \leq x \leq 1/4} G(x) \right)^2 \cdot \left(\sum_{j: \tilde{G}(j) \neq 0} \frac{\tilde{G}(j)^2}{\tilde{K}(j)} \right)^{-1}.$$

The paper [8] uses the parameter $\delta = 0.13$ (thus $u = 0.63$), and the kernel function

$$(7) \quad K(x) = \frac{1}{\delta} \beta \circ \beta \left(\frac{x}{\delta} \right) \quad \text{where } \beta(x) = \frac{2/\pi}{\sqrt{1 - 4x^2}} \quad \left(-\frac{1}{2} < x < \frac{1}{2} \right)$$

(note here that $\|K\|_2^2 < 0.5747/\delta$). Finally, in equation (6) they use one of Selberg's functions, $G(x) = G_{0.63,22}(x)$ defined in Lemma 2.3 of [8]. Combining the statements of Lemma 3.1 above they obtain

$$(8) \quad \|f * f\|_\infty + 1 + \sqrt{\|f * f\|_\infty - 1} \sqrt{\|K\|_2^2 - 1} \geq \\ \geq \frac{2}{u} + 2 \left(\min_{0 \leq x \leq 1/4} G(x) \right)^2 \cdot \left(\sum_{j: \tilde{G}(j) \neq 0} \frac{\tilde{G}(j)^2}{\tilde{K}(j)} \right)^{-1}$$

and substituting the values and estimates they have for u , $\tilde{G}(j)$, $\tilde{K}(j)$, $\min_{0 \leq x \leq 1/4} G(x)$ and $\|K\|_2^2$ the bound $\|f * f\|_\infty \geq 1.262$ follows.

Our improvement of the lower bound on $\|f * f\|_\infty$ comes in two steps. First, we find a better kernel function G in equation (8). This is indeed plausible because Selberg's functions $G_{u,n}$ do not correspond to the specific choice of K in [8] in any way, therefore we can expect an improvement by choosing G so as to minimize the sum $\sum_{j: \tilde{G}(j) \neq 0} \frac{\tilde{G}(j)^2}{\tilde{K}(j)}$, while keeping $\min_{0 \leq x \leq 1/4} G(x) \geq 1$.

Next, we observe that if $\|f * f\|_\infty$ is small then the first Fourier coefficient of f must also be small in absolute value, and we use this information to get a slight further improvement. We will also indicate how the method could yield further improvements.

Theorem 3.2. *If $f : [-\frac{1}{4}, \frac{1}{4}] \rightarrow \mathbb{R}_+$ is a nonnegative function with $\int f = 1$, then $\|f * f\|_\infty \geq 1.2748$.*

Proof. Let $K(x)$ be defined by (7). As in [8] we make use of the facts that $\|K\|_2^2 < 0.5747/\delta$, and $\tilde{K}(j) = \frac{1}{u} |J_0(\pi \delta j/u)|^2$ where J_0 is the Bessel J -function of order 0.

As described above, the main improvement comes from finding a better kernel function G in equation (8). Indeed, if we set $G(x) = \sum_{j=1}^n a_j \cos(2\pi jx/u)$, then $\tilde{G}(j) = \frac{a_{|j|}}{2}$ for $-n \leq j \leq n$ ($j \neq 0$), and thus equation (8) takes the form

$$(9) \quad \|f * f\|_\infty + 1 + \sqrt{\|f * f\|_\infty - 1} \sqrt{0.5747/\delta - 1} \geq \\ \geq \frac{2}{u} + \frac{4}{u} \left(\min_{0 \leq x \leq 1/4} G(x) \right)^2 \cdot \left(\sum_{j=1}^n \frac{a_j^2}{|J_0(\pi \delta j/u)|^2} \right)^{-1}.$$

For brevity of notation let us introduce the “gain-parameter” $a = \frac{4}{u} \left(\min_{0 \leq x \leq 1/4} G(x) \right)^2 \left(\sum_{j=1}^n \frac{a_j^2}{|J_0(\pi \delta j/u)|^2} \right)^{-1}$. We note for the record that $a \approx 0.0342$ for the choices $\delta = 0.13$ and $G(x) = G_{0.63,22}(x)$ in [8]. For

any fixed δ we are therefore led to the problem of maximizing a (while we may as well assume that $\min_{0 \leq x \leq 1/4} G(x) \geq 1$, as G can be multiplied by any constant without changing the gain a). This problem seems hopeless to solve analytically, but one can perform a numerical search using e.g. the “Mathematica 6” software. Having done so, we obtained that for $\delta = 0.138$ and $n = 119$ there exists a function $G(x)$ with the desired properties such that $a > 0.0713$. The coefficients a_j of $G(x)$ are given in the Appendix. Therefore, using this function $G(x)$ and $\delta = 0.138$ in equation (9) we obtain $S \geq 1.2743$.

Remark. One can wonder how much further improvement could be possible by choosing the optimal δ and the optimal $G(x)$ corresponding to it. The answer is that there is *very little* room left for further improvement, the theoretical limit of the argument being somewhere around 1.276. To see this, let $f_s(x) = \frac{1}{2}(f(x) + f(-x))$ denote the symmetrization of f , let $\beta_\delta(x) = \frac{1}{\delta}\beta(\frac{x}{\delta})$ (where $\beta(x)$ is defined in (7)) and reformulate equation (5) as follows:

$$(10) \quad \int (f * f(x) + f \circ f(x)) K(x) dx = 2 \int (f_s * \beta_\delta(x))^2 dx = 2 \|f_s * \beta_\delta\|_2^2.$$

This equality is easy to see using Parseval and the fact that $\tilde{K}(j) = u(\tilde{\beta}_\delta(j))^2$. Now, with $\beta_\delta(x)$ being given, the best lower bound we can possibly hope to obtain for the right hand side is $\inf_{f_s} \|f_s * \beta_\delta\|_2^2$, where the infimum is taken over all nonnegative, symmetric functions f_s with integral 1. To calculate this infimum, one can discretize the problem, i. e. approximate $\beta_\delta(x)$ and $f_s(x)$ by step functions, the heights of the steps of f_s being parameters. Then one can minimize the arising multivariate quadratic polynomial by computer. Finally, we can use equations (3), (4) and (10) to obtain a lower bound for $\|f * f\|_\infty$. We have done this¹ for several values of δ and it seems that best lower bound is achieved for $\delta \approx 0.14$ where we obtain $\|f * f\|_\infty \geq 1.276$. We remark that all this could be done rigorously, but one needs to control the error arising from the discretization, and the sheer documentation of it is simply not worth the effort, in view of the minimal gain.

We can further improve the obtained result a little bit by exploiting some information on the Fourier coefficients of f . For this we need two easy lemmas.

¹The authors are grateful to M. N. Kolountzakis for pointing out that this minimization problem can indeed be solved numerically due to convexity arguments.

Lemma 3.3. *Using the notation $z_1 = |\hat{f}(1)|$ and $k_1 = \hat{K}(1) = \hat{K}(-1)$, where K is defined by equation (7), we have*

$$(11) \quad \int (f \circ f(x))K(x)dx \leq 1 + 2z_1^2k_1 + \sqrt{\|f * f\|_\infty - 1 - 2z_1^4} \sqrt{\|K\|_2^2 - 1 - 2k_1^2}.$$

Proof. This is an obvious modification of Lemma 3.2 in [8]. Namely,

$$\begin{aligned} \int (f \circ f(x))K(x)dx &= \sum_{j \in \mathbb{Z}} (\widehat{f \circ f}(j))\hat{K}(j) \\ &= 1 + 2z_1^2k_1 + \sum_{j \neq 0, \pm 1} |\hat{f}(j)|^2 \hat{K}(j) \\ &\leq 1 + 2z_1^2k_1 + \sqrt{\sum_{j \neq 0, \pm 1} |\hat{f}(j)|^4} \sqrt{\sum_{j \neq 0, \pm 1} \hat{K}(j)^2} \\ &= 1 + 2z_1^2k_1 + \sqrt{\|f * f\|_2^2 - 1 - 2z_1^4} \sqrt{\|K\|_2^2 - 1 - 2k_1^2} \\ &\leq 1 + 2z_1^2k_1 + \sqrt{\|f * f\|_\infty - 1 - 2z_1^4} \sqrt{\|K\|_2^2 - 1 - 2k_1^2}. \end{aligned}$$

□

The next observation is that z_1 must be quite small if $\|f * f\|_\infty$ is small. This is established by an application of the following general fact (the discrete version of which is contained in [4]).

Lemma 3.4. *If h is a nonnegative function with $\int h = 1$, supported on the interval $[-\frac{1}{2}, \frac{1}{2}]$ and bounded above by M , then $|\hat{h}(1)| \leq \frac{M}{\pi} \sin \frac{\pi}{M}$.*

Proof. Observe first that

$$\hat{h}(1) = \int_{\mathbb{R}} h(x)e^{-2\pi i x} dx = e^{-2\pi i t} \int_{\mathbb{R}} h(x+t)e^{-2\pi i x} dx$$

and with a suitable choice of t , the last integral, $\int_{\mathbb{R}} h(x+t)e^{-2\pi i x} dx$, becomes real and nonnegative. Taking absolute values we get

$$|\hat{h}(1)| = \int_{\mathbb{R}} h(x+t) \cos(2\pi x) dx.$$

The lemma becomes obvious now, because in order to maximize this integral, $h(x+t)$ needs to be concentrated on the largest values of the cosine function, so

$$|\hat{h}(1)| \leq \int_{-\frac{1}{2M}}^{\frac{1}{2M}} M \cos(2\pi x) dx = \frac{M}{\pi} \sin \frac{\pi}{M}.$$

□

It is now easy to conclude the proof of Theorem 3.2. Assume $\|f * f\|_\infty < 1.2748$. By Lemma 3.4 we conclude that

$$|\hat{f}(1)| = \sqrt{|\widehat{f * f}(1)|} \leq \sqrt{\frac{1.2748}{\pi} \sin \frac{\pi}{1.2748}} < 0.50426.$$

However, using Lemma 3.3 instead of equation (4) we can replace equation (9) by

$$(12) \quad \begin{aligned} \frac{2}{u} + a &\leq \|f * f\|_\infty + 1 + 2z_1^2 k_1 + \\ &+ \sqrt{\|f * f\|_\infty - 1 - 2z_1^4} \sqrt{0.5747/\delta - 1 - 2k_1^2} \end{aligned}$$

Substituting $\delta = 0.138$, $k_1 = |J_0(\pi\delta)|^2$ and $a = 0.0713$ we obtain a lower bound on $\|f * f\|_\infty$ as a function of z_1 . This function $l(z_1)$ is monotonically decreasing in the interval $[0, 0.50426]$ therefore the smallest possible value for $\|f * f\|_\infty$ is attained when we put $z_1 = 0.50426$. In that case we get $\|f * f\|_\infty = 1.27481$, which concludes the proof of the theorem. $\square$

Remark. In principle, the argument above could be improved in several ways.

First, Lemma 3.4 does not exploit the fact that $h(x)$ is an autoconvolution. It is possible that a much better upper bound on $|\hat{h}(1)|$ can be given in terms of M if we exploit that $h = f * f$.

Second, for any value of $\delta \leq 1/4$ and any suitable kernel functions K and G we obtain a lower bound, $l(z_1)$, for $\|f * f\|_\infty$ as a function of z_1 . A bound $\|f * f\|_\infty \geq s_0$ will follow if z_1 does not fall into the “forbidden set” $F = \{x : l(x) < s_0\}$. In the argument above we put $s_0 = 1.2748$ and, with our specific choices of δ , K and G , the forbidden set was the interval $F = (0.504433, 0.529849)$, and we could prove that z_1 must be outside this set. However, when altering the choices of δ , K and G the forbidden set F also changes. In principle it could be possible that two such sets F_1 and F_2 are disjoint, in which case the bound $\|f * f\|_\infty \geq s_0$ follows automatically.

Third, it is possible to pull out further Fourier coefficients from the Parseval sum in Lemma 3.3, and analyze the arising functions $l(z_1, z_2, \dots)$.

4. COUNTEREXAMPLES

Some papers in the literature conjectured that $S = \pi/2$, with the extremal function being

$$f_0(x) = \frac{1}{\sqrt{2x + 1/2}}, \quad x \in \left(-\frac{1}{4}, \frac{1}{4}\right).$$

Note that $\|f_0 * f_0\|_\infty = \pi/2 = 1.57079\dots$. In particular, the last remark of [9] seems to be the first instance where $\pi/2$ is suggested as the extremal value, while the recent paper [8] includes this conjecture explicitly as Conjecture 5.1. In this section we disprove this conjecture by means of specific examples. The down side of such examples, however, is that we do not arrive at any reasonable new conjecture for the true value of S or the extremal function where it is attained.

The results of this section are produced by computer search and we do not consider them deep mathematical achievements. However, we believe that they are important contributions to the subject, mostly because they can save considerable time and effort in the future to be devoted to the proof of a natural conjecture which is in fact false. We also emphasize here that although we disprove the conjectures made in [9] and in [8], this does not reduce the value of the main results of those papers in any way.

The counterexamples are produced by a computer search. This is most conveniently carried out in the discretized version of the problem. That is, we take an integer n and consider only nonnegative step functions which take constant values a_j on the intervals $[-\frac{1}{4} + \frac{j}{2n}, -\frac{1}{4} + \frac{j+1}{2n})$ for $j = 0, 1, \dots, n-1$. This is equivalent to considering all the nonzero polynomials $P(x) = a_0 + a_1x + \dots + a_{n-1}x^{n-1}$ with nonnegative coefficients such that $\sum_{j=0}^{n-1} a_j = \sqrt{2n}$ and their squares $P^2(x) = b_0 + b_1x + \dots + b_{2n-2}x^{2n-2}$, and asking for the infimum of the maximum of the b_j 's. Schinzel and Schmidt proved [9] that this value is $\geq S$ and its limit when $n \rightarrow \infty$ is S .

Note 4.1. Our constant S can also be defined as $S = \inf_{g \in \mathcal{G}} \frac{\|f * f\|_\infty}{\|f\|_2^2}$ where $\mathcal{G}$ is the set of all nonnegative real functions g , not identically 0, supported on the interval $[-\frac{1}{4}, \frac{1}{4}]$.

The same thing happens in the discrete version. We can consider the set $\mathcal{P}$ of all nonzero polynomials of degree $\leq n-1$ with nonnegative real coefficients $P(x) = a_0 + a_1x + \dots + a_{n-1}x^{n-1}$ and their squares

$P^2(x) = b_0 + b_1x + \cdots + b_{2n-2}x^{2n-2}$ and ask for the value of

$$(13) \quad 2n \inf_{P \in \mathcal{P}} \frac{\max_j b_j}{\left(\sum_{j=0}^{n-1} a_j\right)^2},$$

and we will obtain the same value S as before.

Although our examples will be “normalized” in order to fit the first definitions (i.e. all integrals will be normalized to 1, and all sums will be normalized to $\sqrt{2n}$), most of the computations we have been carried out using these other ones (which are more convenient and closer to the ones given by Schinzel and Schmidt). This note also justifies the fact that it is not a problem if we have an integral which is not exactly equal to 1 or a sum of coefficients in a polynomial which is not exactly equal to $\sqrt{2n}$ because of small numerical errors.

While we can only search for *local* minima numerically, using the “Mathematica 6” software we have been able to find examples of step functions with $\|f * f\|_\infty < 1.522$, much lower than $\pi/2$. Subsequently, better examples were produced with the LOQO solver (Student version for Linux and on the NEOS server²), reaching the value $\|f * f\|_\infty = 1.51237\dots$. The best example we are currently aware of has been produced by an iterative algorithm designed by M. N. Kolountzakis and the first author. The idea is as follows: take any step function $f = (a_0, a_1, \dots, a_{n-1})$ as a starting point, normalized so that $\sum a_j = \sqrt{2n}$. By means of linear programming it is easy (and quick) to find the step function $g_0 = (b_0, b_1, \dots, b_{n-1})$ which maximizes $\sum b_j$ while keeping $\|f * g_0\|_\infty \leq \|f * f\|_\infty$ (obviously, $\sum b_j \geq \sqrt{2n}$ because the choice $g_0 = f$ is legitimate). We then re-normalize g_0 as $g = \frac{\sqrt{2n}g_0}{\sum b_j}$. Then $\|f * g\|_\infty \leq \|f * f\|_\infty$ by construction. If the inequality is strict then it is easy to see that for small $t > 0$ the function $h = (1-t)f + tg$ will be better than our original f , i. e. $\|h * h\|_\infty \leq \|f * f\|_\infty$. And we iterate this procedure until a fix-point function is reached.

The best example produced by this method is included in the Appendix, achieving the value $\|f * f\|_\infty = 1.50972\dots$. Figure 1 shows a plot of the autoconvolution of this function.

Interestingly, it seems that the smallest value of n for which a counterexample exists is as low as $n = 10$, giving the value 1.56618... We include the coefficients of one of these polynomials here, as it is fairly easy to check even by hand:

²We are grateful to Imre Barany and Robert J. Vanderbei who helped us with a code for LOQO.

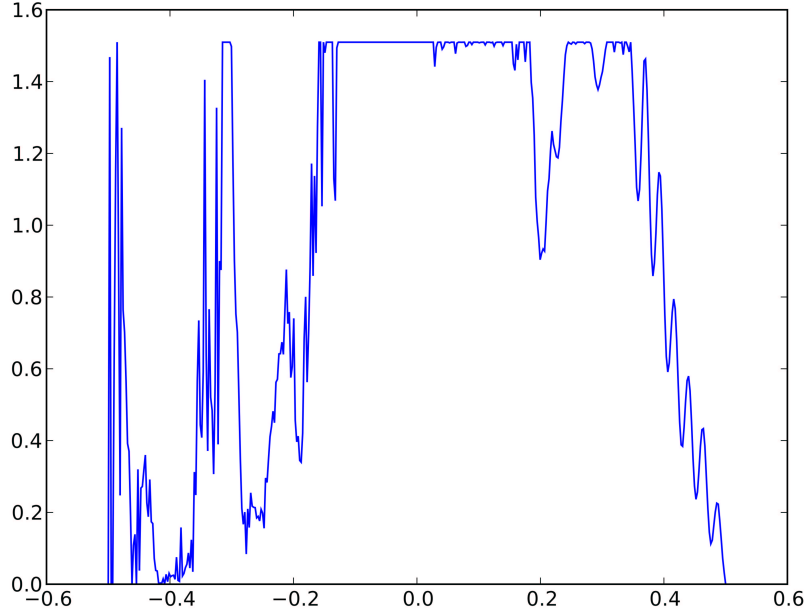


FIGURE 1. The autoconvolution of the best step function we are aware of, giving $\|f * f\|_\infty = 1.50972\dots$

0.41241661	0.45380115	0.51373388	0.6162143	0.90077119
0.14003277	0.16228556	0.19989487	0.2837527	0.78923292

The down side of such examples is that it seems virtually impossible to guess what the extremal function might be. We have looked at the plot of many step functions f with integral 1 and $\|f * f\|_\infty < 1.52$ and several different patterns seem to arise, none of which corresponds to an easily identifiable function. Looking at one particular pattern we have been able to produce an analytic formula for a function f which gives a value for $\|f * f\|_\infty \approx 1.52799$, comfortably smaller than $\pi/2$ but which is somewhat far from the minimal value we have achieved with step functions. This function f is given as:

$$(14) \quad f(x) = \begin{cases} \frac{1.392887}{(0.00195 - 2x)^{1/3}} & \text{if } x \in (-1/4, 0) \\ \frac{0.338537}{(0.500166 - 2x)^{0.65}} & \text{if } x \in (0, 1/4) \end{cases}$$

Figure 2 shows a plot of the autoconvolution of this function.

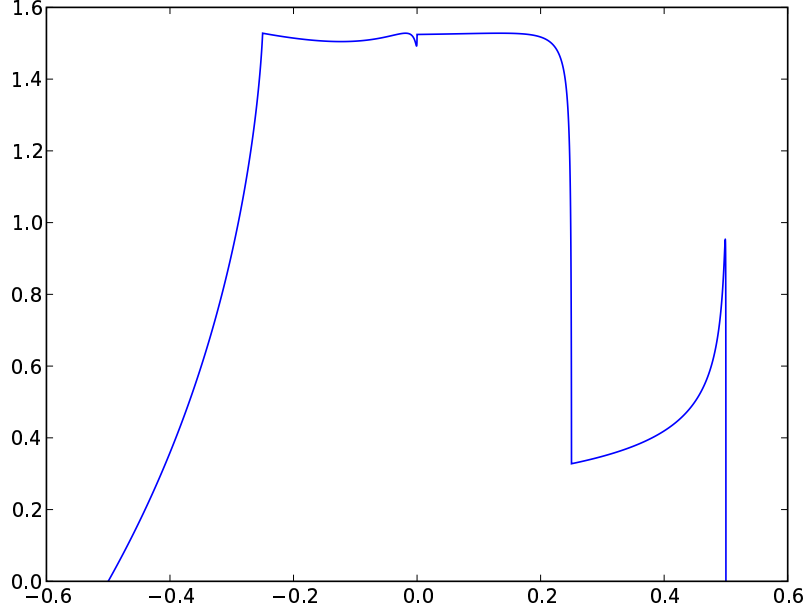


FIGURE 2. The autoconvolution of the function given by equation (14), giving $\|f * f\|_\infty \approx 1.52799$.

The paper [8] also states in Conjecture 2 that an inequality of the form

$$(15) \quad \|f * f\|_2^2 \leq c \|f * f\|_\infty \|f * f\|_1$$

should be true with the constant $c = \frac{\log 16}{\pi}$, and once again the function f_0 above producing the extremal case. While we tend to believe that such an inequality is indeed true with some constant $c < 1$, we have been able to disprove this conjecture too, and find examples where $c > \frac{\log 16}{\pi}$. We have not made extensive efforts to maximize the value of c in our numerical search. In the Appendix we include one example of a step function with $n = 20$ where $c = 0.88922... > \frac{\log 16}{\pi} = 0.88254...$

We make a last remark here that could be of interest. It is somewhat natural to believe that the minimal possible value of $\|f * f\|_\infty$ does not change if we allow f to take negative values (but keeping $\int f = 1$). However, this does not seem to be the case. We have found examples of step functions f for which $\|f * f\|_\infty = 1.45810...$, much lower than the best value ($\|f * f\|_\infty = 1.50972...$) we have for nonnegative functions f . This example is also included in the Appendix.

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APPENDIX (ONLINE VERSION ONLY)

Here we list the numerical values corresponding to the results of the previous sections.

For $\delta = 0.138$ (and thus $u = 0.638$) we define the kernel function $G(x)$ used in Theorem 3.2 as $G(x) = \sum_{j=1}^{119} a_j \cos(2\pi jx/u)$, with the coefficients a_j given by the following list:

2.16620392e+00	-1.87775750e+00	1.05828868e+00	-7.29790538e-01
4.28008515e-01	2.17832838e-01	-2.70415201e-01	2.72834790e-02
-1.91721888e-01	5.51862060e-02	3.21662512e-01	-1.64478392e-01
3.95478603e-02	-2.05402785e-01	-1.33758316e-02	2.31873221e-01
-4.37967118e-02	6.12456374e-02	-1.57361919e-01	-7.78036253e-02
1.38714392e-01	-1.45201483e-04	9.16539824e-02	-8.34020840e-02
-1.01919986e-01	5.94915025e-02	-1.19336618e-02	1.02155366e-01
-1.45929982e-02	-7.95205457e-02	5.59733152e-03	-3.58987179e-02
7.16132260e-02	4.15425065e-02	-4.89180454e-02	1.65425755e-03
-6.48251747e-02	3.45951253e-02	5.32122058e-02	-1.28435276e-02
1.48814403e-02	-6.49404547e-02	-6.01344770e-03	4.33784473e-02
-2.53362778e-04	3.81674519e-02	-4.83816002e-02	-2.53878079e-02
1.96933442e-02	-3.04861682e-03	4.79203471e-02	-2.00930265e-02
-2.73895519e-02	3.30183589e-03	-1.67380508e-02	4.23917582e-02
3.64690190e-03	-1.79916104e-02	7.31661649e-05	-2.99875575e-02
2.71842526e-02	1.41806855e-02	-6.01781076e-03	5.86806100e-03
-3.32350597e-02	9.23347466e-03	1.47071722e-02	-7.42858080e-04
1.63414270e-02	-2.87265671e-02	-1.64287280e-03	8.02601605e-03
-7.62613027e-04	2.18735533e-02	-1.78816282e-02	-6.58341101e-03
2.67706547e-03	-6.25261247e-03	2.24942824e-02	-8.10756022e-03
-5.68160823e-03	7.01871209e-05	-1.15294332e-02	1.83608944e-02
-1.20567880e-03	-3.13147456e-03	1.39083675e-03	-1.49312478e-02
1.32106694e-02	1.73474188e-03	-8.53469045e-04	4.03211203e-03
-1.55352991e-02	8.74711543e-03	1.93998895e-03	-2.71357322e-05
6.13179585e-03	-1.41983972e-02	5.84710551e-03	9.22578333e-04
-2.16583469e-04	7.07919829e-03	-1.18488582e-02	4.39698322e-03
-8.91346785e-05	-3.42086367e-04	6.46355636e-03	-8.87555371e-03
3.56799654e-03	-4.97335419e-04	-8.04560326e-04	5.55076717e-03
-7.13560569e-03	4.53679038e-03	-3.33261516e-03	2.35463427e-03
2.04023789e-04	-1.27746711e-03	1.81247830e-04	

The best nonnegative step function we are currently aware of, reaching the value $\|f * f\|_\infty = 1.50972\dots$, is attained at $n = 208$. The coefficients of its associate polynomial (a polynomial of degree 207 whose coefficients sum up to $\sqrt{416}$) are:

1.21174638	0.	0.	0.25997048	0.47606812
0.62295219	0.3296586	0.	0.29734381	0.
0.	0.	0.	0.	0.
0.	0.00846453	0.05731673	0.	0.13014906
0.	0.08357863	0.05268549	0.06456956	0.06158231
0.	0.	0.	0.	0.
0.	0.	0.	0.	0.
0.	0.	0.	0.	0.
0.	0.	0.	0.	0.
0.02396999	0.	0.	0.05846552	0.
0.	0.	0.	0.	0.0026332
0.0509835	0.	0.1283313	0.0904924	0.21232176
0.24866151	0.09933512	0.01963586	0.01363895	0.32389841
0.	0.	0.14467517	0.0129752	0.
0.	0.16299837	0.38329665	0.11361262	0.32074656
0.17344291	0.33181372	0.24357561	0.2577003	0.20567824
0.13085743	0.17116496	0.14349025	0.07019695	0.
0.	0.	0.	0.	0.
0.	0.	0.	0.	0.
0.	0.	0.	0.	0.
0.	0.0131741	0.0342541	0.0427565	0.03045044
0.07900079	0.07020678	0.08528342	0.09705597	0.0932896
0.09360206	0.06227754	0.07943462	0.08176106	0.10667185
0.10178412	0.11421821	0.07773213	0.11021377	0.12190377
0.06572457	0.07494855	0.	0.	0.02140202
0.	0.	0.0231478	0.00127997	0.
0.04672881	0.03886266	0.11141784	0.00695668	0.0466224
0.03543131	0.08803511	0.04165729	0.10785652	0.06747342
0.18785215	0.31908323	0.3249705	0.09824861	0.23309878
0.12428441	0.03200975	0.0933163	0.09527521	0.12202693
0.13179059	0.09266878	0.02013746	0.16448047	0.20324945
0.21810431	0.27321179	0.25242816	0.19993811	0.13683837
0.13304836	0.08794214	0.12893672	0.16904485	0.22510883
0.26079786	0.27367504	0.26271896	0.20457964	0.15073917
0.11014028	0.09896	0.0926069	0.13269111	0.17329988
0.20761774	0.21707182	0.18933169	0.14601258	0.08531506
0.06187865	0.06100211	0.09064962	0.12781018	0.17038096
0.185766	0.1734501	0.14667009	0.09569536	0.06092822
0.03219067	0.0495587	0.09657756	0.16382398	0.22606693
0.22230709	0.19833621	0.16155032	0.09330751	0.02838363
0.02769322	0.03349924	0.09448887	0.20517242	0.22849741
0.24175836	0.19700135	0.18168723		

The best example of a step function disproving Conjecture 2 of [8], we are currently aware of, is attained for $n = 20$ (note that we did not make extensive efforts to optimize this example).

1.27283	0.54399	0.	0.	0.	0.
0.	0.529367	0.410195	0.46111	0.439352	0.448675
0.444699	0.446398	0.335601	0.322369	0.240811	0.202225
0.138305	0.0886248				

This function reaches the value $c = 0.88922... > \frac{\log 16}{\pi}$ in equation (15).

Finally, the best step function we are currently aware of (which takes some negative values!), reaching the value $\|f * f\|_{\infty} = 1.45810...$, is attained at $n = 150$. The coefficients of its associate polynomial are:

0.7506545	0.4648332	0.59759775	0.46028561	0.36666088
0.37773841	0.16162776	0.3303943	0.15905831	0.08878588
0.16284952	-0.09198076	0.05755583	-0.00690908	-0.08627636
-0.17180424	-0.14778207	0.13121791	0.05268415	0.20694965
0.25287625	0.2071192	-0.13591836	0.05354584	-0.03558645
0.15699341	-0.06508942	-0.01435246	0.02291645	0.18877783
-0.02751401	0.09592962	0.06666674	0.1807308	0.15543041
0.02639022	0.01843893	0.04896963	0.0303207	0.05119754
0.24099308	0.2244329	0.23689694	0.08980581	0.25272138
0.26725296	0.12786816	0.16265063	0.20542404	0.06826679
0.16905985	-0.11230055	0.26179213	-0.412312	-0.28820566
-0.7619902	-0.78933468	0.07066217	0.05785475	0.07163788
0.09949514	0.0659708	0.05370837	0.08441868	0.10157278
0.07317574	0.0521853	0.08980666	0.13113512	0.05943309
0.07517572	0.12460218	0.14885796	0.09071907	0.13017884
0.13185969	0.15196722	0.07848544	0.14924624	0.16053609
0.17735544	0.14470971	0.17275872	0.16058981	0.22807136
0.20728811	0.10876597	0.21471959	0.25136905	0.15147268
0.06366331	0.05917714	0.05995267	0.35288009	0.3224057
0.32988077	0.41806458	0.22880318	0.2080819	0.18504847
0.27116284	0.16066195	0.02547032	0.26150045	-0.00634039
0.09471136	-0.00407705	0.04759596	-0.07549638	-0.30815721
-0.00878173	0.08964445	0.23265916	0.37008611	0.18283593
0.00240797	0.063899	0.02892268	0.10802879	0.15672677
-0.11335258	0.10549109	0.1571762	0.13290998	-0.01251118
0.15487122	0.15770952	0.33037764	0.03888211	0.08105707
0.00799348	0.00375632	-0.02392944	0.15019215	0.21615677
0.17854093	0.04104506	0.12700956	0.23964236	0.05613369
0.14857745	0.07375734	0.02816608	0.16226977	0.01757525
-0.23848002	0.05705152	0.29372066	0.56730329	1.105205

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M.M.: ALFRÉD RÉNYI INSTITUTE OF MATHEMATICS, REALTANODA U 13-15, BUDAPEST, HUNGARY (ALSO AT BME DEPARTMENT OF ANALYSIS, BUDAPEST, H-1111, EGRY J. U. 1).

E-mail address: `matomate@renyi.hu`

C.V.: UNIVERSIDAD AUTÓNOMA DE MADRID, CIUDAD UNIVERSITARIA DE CANTOBLANCO, MADRID, SPAIN.

E-mail address: `c.vinuesa@uam.es`