Smoothness of the stable foliation versus decay of correlations for Lorenz-like attractors

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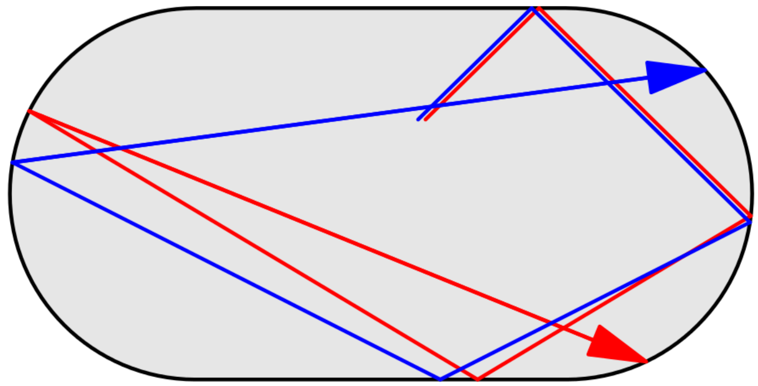
Contents

1 Setting

Chaotic flows



Continuous time. Physical/SRB measure.

M compact metric space

$X^t : M \to M$ smooth flow (ie, $X^{t+s} = X^t \circ X^s$ for $s, t \in \mathbb{R}$)

There exists a unique invariant SRB measure μ_{SRB}: $Leb(B(\mu_{SRB})) > 0$ where

$$B(\mu_{SRB}) = \left\{ x \in M : \frac{1}{T} \int_0^T \varphi(X^s(x))\, ds \to \int \varphi \, d\mu_{SRB}, \quad \forall \varphi \in C(M) \right\}$$

1.1 Mixing

Mixing

After obtaining an interesting invariant probability measure for a dynamical system, it is natural to study the properties of this measure. Besides ergodicity there are various degrees of mixing.

Given a flow X^t and an invariant ergodic probability measure μ, we say that the system (X^t, μ) is ***mixing*** if for any two measurable sets A, B

$$\mu\big(A \cap X^{-t} B\big) \xrightarrow[t \to \infty]{} \mu(A) \cdot \mu(B)$$

or equivalently

$$\int \varphi \cdot \big(\psi \circ X^t\big)\, d\mu \xrightarrow[t \to \infty]{} \int \varphi \, d\mu \int \psi \, d\mu$$

for any pair $\varphi, \psi : M \to \mathbb{R}$ of continuous functions.

Correlation function

Considering φ and $\psi \circ X^t : M \to \mathbb{R}$ as random variables over the probability space (M, μ), this definition just says that "**the random variables φ and $\psi \circ X^t$ are asymptotically independent**" since the expected value $\mathbb{E}\big(\varphi \cdot (\psi \circ X^t)\big)$ tends to the product $\mathbb{E}(\varphi) \cdot \mathbb{E}(\psi)$ when t goes to infinity.

The ***correlation function***

$$C_t(\varphi, \psi) = \big|\mathbb{E}\big(\varphi \cdot (\psi \circ X^t)\big) - \mathbb{E}(\varphi) \cdot \mathbb{E}(\psi)\big| \\ = \Big| \int \varphi \cdot (\psi \circ X^t)\, d\mu - \int \varphi\, d\mu \int \psi\, d\mu \Big|$$

satisfies $C_t(\varphi, \psi) \xrightarrow[t\to\infty]{} 0$ in the case of mixing.

Speed of mixing: decay of correlations.

Given observables $\varphi, \psi : M \to \mathbb{R}$ in a Banach space X (which depends on the systems and is in general a space of functions with some regularity, Hölder or C^r for some $r > 1$...) the correlation function (for the SRB measure) is given by

$$C_t(\psi, \varphi) = \left| \int (\varphi \circ X^t)\, \psi\, d\mu_{SRB} - \int \psi\, d\mu_{SRB} \int \varphi\, d\mu_{SRB} \right|.$$

We classify decay of correlations into some classes

- Exponential decay: $\exists C, \gamma > 0$ so that

$$C_t(\psi, \varphi) \leq Ce^{-\gamma t} \|\psi\| \|\varphi\|$$

- Super-polynomial decay: $\forall \beta > 0 \exists C_\beta > 0$ s.t.

$$C_t(\psi, \varphi) \leq C_\beta t^{-\beta} \|\psi\| \|\varphi\|$$

1.2 Overview

Some known results: Decay of correlations

	super-poly. decay	exp. decay
Anosov or Axiom A flows	C^2 open C^∞ dense **not all**	smooth foliations & non-integrability C^1 open set of C^3 dim≥ 3 vector fields
geometric Lorenz attractors	C^2 open C^∞ dense	C^1-open set of C^∞ vector fields

Dolgopyat 98' C^5-Anosov flows whose stable and unstable foliations are *jointly non-integrable* have exponential decay

Dolgopyat 98' Generic suspension flows over subshift of finite type are exponentially mixing

Pollicott 99' Equilibrium states of suspension semiflows over subshift of finite type with "nice" roof function have exponential decay

Field, Melbourne, Török 07' C^2 open, C^∞ dense set of Axiom A flows with *super-polynomial* decay of correlations

Ruelle 83', Pollicott 85' Examples with slow decay of correlations.

Baladi, Vallée 05' Exp. decay of corr. for C^2 suspension semiflows on surfaces with countable Markov partitions and "good roof function"

Ávila, Goüezel, Yoccoz 06' Exponential decay of correlations for Teichmüller flow; criterium for suspension semiflows over hyperbolic base with (countable) Markov structure

Melbourne 09' C^2 open, C^∞ dense set of geom. Lorenz attractors have superpolynomial decay

A., Varandas 11' C^2-open set of geom. Lorenz attractors with exponential decay

A., Melbourne, Varandas 15' Super-polynomial decay for C^1 open set of C^∞ geometric Lorenz attractors and ASIP for time-1 map

A., Butterley, Varandas 16' C^1-open set C^3 Axiom A vector fields, dim. ≥ 3, with non-trivial attractor with exponential decay

A., Melbourne 16' Exponential decay of correlations for $C^{1+\alpha}$ suspension semiflows on surfaces with countable Markov partitions and "good roof function"

1.3 Geometric Lorenz flow and exponential decay

Lorenz equations

In 1963 Lorenz presented the following systems of equations and payed close attention to certain parameter values:

$$\frac{dx}{dt} = \sigma(y-x) \qquad \sigma = 10$$
$$\frac{dy}{dt} = rx - y - xz \qquad r = 28$$
$$\frac{dz}{dt} = xy - bz \qquad b = 8/3$$

for which the systems seemed to be “sensitive to initial conditions” or “chaotic”.

The Lorenz system has an attractor

Only around the year 2000 was it established, by [Tucker, "The Lorenz attractor exists", C. R. Acad. Sci. Paris, 1999], that **the Lorenz system of equations with the parameters indicated by Lorenz does indeed have a transitive attractor with a SRB measure.**

This proof was and remains a computer assisted proof, rather involved, delicate and quite technical, which works for a specific family of parameters. It was tested on very fast computers at the time and took several days to complete the calculations.

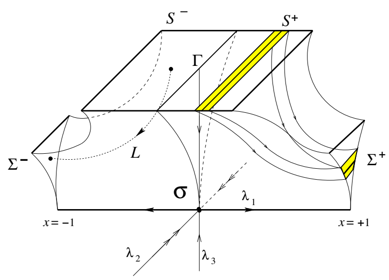
Tucker in fact showed that the Lorenz attractor is a geometric Lorenz attractor, and so is an example of transitive singular-hyperbolic set

Description of Geometric Lorenz attractors

Consider the linear system $(\dot{x},\dot{y},\dot{z}) = (\lambda_1 x, \lambda_2 y, \lambda_3 z)$, thus

$$X^t(x_0,y_0,z_0) = (e^{\lambda_1 t}x_0, e^{\lambda_2 t}y_0, e^{\lambda_3 t}z_0),$$

where $\lambda_2 < \lambda_3 < 0 < -\lambda_3 < \lambda_1$ in a ngbh. of $(0,0,0)$.



For $\tau = -\frac{1}{\lambda_1}\log|x|$ we get

$$X^\tau(x,y,1) = (\operatorname{sgn}(x), y|x|^{-\lambda_2/\lambda_1}, |x|^{-\lambda_3/\lambda_1})$$

Invariant constracting foliation

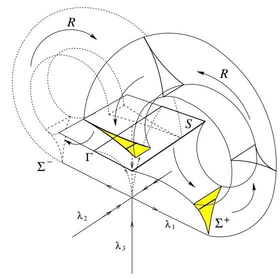
We assume that the "triangles" $L(S^\pm)$ are compressed in the y-direction and stretched on the other transverse and rotated back **preserving the line segments** $S \cap \{x = x_0\}$: This may be seen as a suspension flow over the Poincaré return map R with roof function $\tau_X(x,y) = -\frac{1}{\lambda_1}\log|x| + c(x)$ where $c(\cdot)$ is bounded.

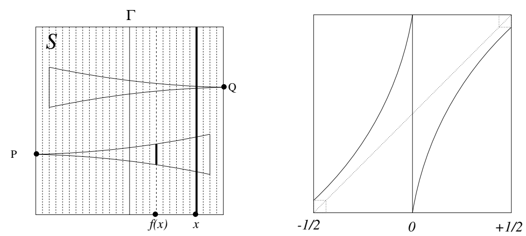
One-dimensional quotient map

The Poincaré first return map $R : S^* \to S$ is a skew-product $R(x,y) = \big(f(x), g(x,y)\big)$ for some functions $f : I \setminus \{0\} \to I$ and $g : (I \setminus \{0\}) \times I \to I$, where $I = [-1/2, 1/2]$.

Moreover, **the smoothness of f depends on the smoothness of the contracting foliation** and

- $f(x) \approx |x|^\alpha$ and so $|f'(x)| \approx \alpha|x|^{\alpha-1}$
- $|\partial_y g| \approx |x|^\beta \le \lambda < 1$ and $|\partial_y g(x,y)| \cdot |Df(x)|^{-1} \le \lambda$ which give singular-hyperbolicity for the attrator.





Sectional-hyperbolicity

Tucker in fact showed that the Lorenz attractor is a transitive singular-hyperbolic set.

We say that a compact invariant set Λ for a flow is ***sectionally hyperbolic*** if the tangent bundle over Λ admits a DX_t-invariant and **dominated splitting** $T_\Lambda M = E^s_\Lambda \oplus E^c_\Lambda$, such that there are $C, \lambda > 0$ satisfying for every $x \in \Lambda$ and $t > 0$

- E^s is uniformly contracted: $\|DX_t \mid E^s_x\| \leq Ce^{-\lambda t}$;
- E^c is 2-sectionally expanded: for every bidimensional subspace F_x contained in E^c_x we have $|\det(DX_t \mid F_x)| \geq Ce^{\lambda t}$; and
- all equilibrium points, if any, are hyperbolic.

Sectional-hyperbolicity and hyperbolicity

A sectional-hyperbolic compact invariant subset for a three-dimensional vector field (where $\dim E^s = 1$ and $\dim E^c = 2$) is also referred to as a ***singular-hyperbolic*** **set.**

Sectional-hyperbolicity is an extension of the notion of hyperbolicity.

Hyperbolic Lemma

Every compact singular-hyperbolic set without singularities is an hyperbolic set, that is, E^c can be written as $[G] \oplus E^u$, where $[G]$ is the flow direction and E^u is uniformly expanded:

$$\exists C, \lambda > 0 : \|(DX_t \mid E^u_x)^{-1}\| < Ce^{-\lambda t}.$$

Dominated splitting. Robustness.

The continuous splitting $T_\Lambda M = E^s \oplus E^c$ is **dominated** if it is DX_t-invariant, that is

$$DX_t E^*_x = E^*_{X^t(x)}, \forall t \in \mathbb{R}, \forall x \in \Lambda, * = s, cu;$$

and there are $K, \lambda > 0$ such that

$$\|DX_t \mid E_x\| \cdot \|DX_{-t} \mid E^c_{X_t(x)}\| < Ke^{-\lambda t}, \forall\, x \in \Lambda, t > 0.$$

Domination is a rather weak form of hyperbolicity, but is a robust property. This means that if a vector field Z admits an attracting set Λ, then there we can find $\varepsilon > 0$ such that for all vector fields Y such that $\|Y - Z\|_{C^1} < \varepsilon$ there is an attracting set Λ_Y close to Λ so that Λ_Y has a dominated splitting (with the same dimensions of the subbundles).

This robustness property is also true for sectional hyperbolicity.

The stable (contracting) foliation

To construct the physical/SRB measure for a geometric Lorenz attractor the smoothness of the one-dimensional quotient map is important: it needs to be a $C^{1+\alpha}$ piecewise expanding map with finitely many branches, for some $\alpha > 0$.

This crucially depends on the regularity of the contracting foliation over which the dynamics of the return map is quotiented.

Moreover, the construction of geometric Lorenz attractors provides that **this contracting foliation covers a full neighborhood U of the attractor.**

The attractor has zero volume

Moreover, the Lorenz equations define a vector field G which is dissipative, that is, $\operatorname{div}(G) \leq -\delta < 0$ for some $\delta > 0$.

Hence, the Lorenz attractor $\Lambda = \overline{\cap_{t>0} X_t(U)}$ has zero volume, where X_t is the flow generated by G.

However, this is a general result: a singular-hyperbolic attractor has zero volume whenever the vector field is of class $C^{1+\alpha}$ [see Alves, A., Pacifico, Pinheiro, Dyn Sist an Int J, **22**(3), 249-267 (2007)].

Strongly dissipative condition

We further assume that our geometric Lorenz flows are ***strongly dissipative***, i.e., the divergence of the vector field G is strictly negative: there exists $\delta > 0$ such that

$$(\operatorname{div} G)(x) \leq -\delta, \quad \forall x \in U,$$

and moreover the eigenvalues of the singularity at 0 satisfy the additional constraint

$$\lambda_u + \lambda_{ss} < \lambda_s \quad (\lambda_1 + \lambda_2 < \lambda_3).$$

A consequence of domination, uniform contraction on the stable direction and strong dissipativity, is the existence of a X_t-invariant contracting foliation $\mathcal{F}^{ss}$, defined in a neighborhood of Λ, which is $C^{1+\varepsilon}$-smooth and whose leaves are $C^{1+\varepsilon}$ curves with uniform size.

Lemma 1. *The strong stable foliation $\mathcal{F}^{ss}$ is $C^{1+\varepsilon}$ for some $\varepsilon > 0$.*

1.4 Sketch

Sketch of proof of exponential decay

To obtain exponential decay for geometric Lorenz flow the strategy is to show that this flow can be written as a **semiflow over** $C^{1+\alpha}$ **expanding maps with** C^1 **roof functions satisfying a uniform non-integrability condition**. We now explain the terms,

Uniformly expanding maps:

Fix $\alpha \in (0,1]$. Let $\{(c_m, d_m) : m \geq 1\}$ be a countable partition mod 0 of $Y = [0,1]$ and suppose that $F : Y \to Y$ is $C^{1+\alpha}$ on each subinterval (c_m, d_m) and extends to a homeomorphism from $[c_m, d_m]$ onto Y.

Let $\mathcal{H} = \{h : Y \to [c_m, d_m]\}$ denote the family of inverse branches of F, and let $\mathcal{H}_n$ denote the inverse branches for F^n.

Uniformly expanding maps and absolutely continuous invariant probability measures

We say that $F : Y \to Y$ is a $C^{1+\alpha}$ ***uniformly expanding map*** if there exist constants $C_1 \geq 1$, $\rho_0 \in (0,1)$ s.t.

(i) $|h'|_\infty \leq C_1 \rho_0^n$ for all $h \in \mathcal{H}_n$,

(ii) $|\log|h'|\,|_\alpha \leq C_1$ for all $h \in \mathcal{H}$,

where

$$|\log|h'|\,|_\alpha = \sup_{x \neq y} \frac{\big|\log|h'|(x) - \log|h'|(y)\big|}{|x-y|^\alpha}.$$

Under these assumptions, it is standard that there exists a unique F-invariant absolutely continuous probability measure μ with α-Hölder density bounded above and below.

Expanding semiflows

Suppose that $R : Y \to \mathcal{R}^+$ is C^1 on partition elements (c_m, d_m) with $\inf R > 0$. Define the suspension $Y^R = \{(y,u) \in Y \times \mathcal{R} : 0 \leq u \leq R(y)\}/\sim$ where $(y, R(y)) \sim (Fy, 0)$.

The suspension flow $F_t : Y^R \to Y^R$ is given by $F_t(y,u) = (y, u+t)$ computed modulo identifications, with ergodic invariant probability measure $\mu^R = (\mu \times \mathrm{Leb})/\bar{R}$ where $\bar{R} = \int_Y R\, d\mu$.

We say that F_t is a $C^{1+\alpha}$ ***expanding semiflow*** provided

(iii) $|(R\circ h)'|_\infty \le C_1$ for all $h\in\mathcal{H}$.

(iv) There exists $\varepsilon>0$ such that $\sum_{h\in\mathcal{H}} e^{\varepsilon|R\circ h|_\infty}|h'|_\infty<\infty$.

Uniform nonintegrability

Let $R_n=\sum_{j=0}^{n-1} R\circ F^j$ and define

$$\psi_{h_1,h_2}=R_n\circ h_1-R_n\circ h_2 : Y\to\mathcal{R},$$

for $h_1,h_2\in\mathcal{H}_n$. We require

(UNI) There exists $D>0$, and $h_1,h_2\in\mathcal{H}_{n_0}$, for some sufficiently large integer $n_0\ge 1$, such that $\inf|\psi'_{h_1,h_2}|\ge D$.

The requirement "sufficiently large" can be made explicit.

Function spaces

Define $F_\alpha(Y^R)$ to consist of L^∞ functions $v:Y^R\to\mathcal{R}$ such that $\|v\|_\alpha=|v|_\infty+|v|_\alpha<\infty$ where

$$|v|_\alpha=\sup_{(y,u)\neq(y',u)}\frac{|v(y,u)-v(y',u)|}{|y-y'|^\alpha}.$$

Define $F_{\alpha,k}(Y^R)$ to consist of functions with $\|v\|_{\alpha,k}=\sum_{j=0}^k\|\partial_t^j v\|_\alpha<\infty$ where ∂_t denotes differentiation along the semiflow direction.

Obtaining exponential decay

Given $v,w\in F_{\alpha,1}(Y^R)$ define the correlation function

$$\rho_{v,w}(t)=\int v\,w\circ F_t\,d\mu^R-\int v\,d\mu^R\int w\,d\mu^R.$$

Theorem [Baladi-Vallée '05 (with C^2 expanding map), A.-Melbourne '15 (with $C^{1+\alpha}$ expanding map)]

Assume conditions (i)–(iv) and UNI. Then there exist constants $c,C>0$ s.t. for all $t>0$

$$|\rho_{v,w}(t)|\le Ce^{-ct}\|v\|_{\alpha,2}\|w\|_{\alpha,2}.$$

1.5 Organization of the notes and talks

Plan of the talks: stable bundle and foliation

Assume that Λ is an attracting set with a continuous invariant partially hyperbolic splitting $T_\Lambda=E^s\oplus E^{cu}$: we have domination plus E^s uniformly contracted. We get

- a positively invariant ngbh. U_0 of Λ and a continuous family of cone fields $\mathcal{C}^s(a)$, $\mathcal{C}^{cu}(a)$ over U_0 satisfying backwards expansion of $\mathcal{C}^s(a)$ and domination.
- a continuous extension of the stable subspace bundle E^s over Λ to an *invariant contracting bundle E^s over U_0*.

- a flow invariant contracting stable manifold bundle W^s over U_0 consisting of C^1 leaves tangent to E^s, which is a topological foliation of U_0.

Then we study the smoothness of this foliation.

Plan of the talks: smoothness of stable foliation

- Show that bunching implies smoothness of the stable foliation W^s, with the regularity being at least Hölder, and the holonomies along this foliation have the same regularity.

In addition to the previous assumptions, assume sectional expansion on E^{cu}.

- Then strong dissipativity implies regularity, as in the previous item.

Assume, in addition, that E^s has codimension 2.

- Then the quotient one-dimensional map is $C^{1+\varepsilon}$ (even though the stable foliation is only Hölder regular).

Finally, assuming also sectional expansion on E^{cu}

- Then the one-dimensional quotient map is a piecewise $C^{1+\varepsilon}$ expanding map.

2 Stable bundle

Invariant stable bundle extension

Existence of an invariant extension of the stable bundle to a full neighborhood of the attracting set

We discuss existence and regularity properties of the stable foliation associated with a partially hyperbolic attracting set. **Sectional expansion is not assumed.**

Throughout, Λ is a partially hyperbolic attractor for a vector field $G \in \mathfrak{X}^r(M)$, $r \geq 1$, with dominated invariant splitting $T_\Lambda M = E^s \oplus E^{cu}$ and E^s uniformly contracted. Write $d = \dim M = d_s + d_{cu}$.

2.1 Cone fields

Cone fields in a neighborhood of Λ

Let $U_0 \subset M$ be a forward invariant neighborhood of Λ such that $\bigcap_{t\geq 0} X_t(U_0) = \Lambda$.

Choose a continuous (not necessarily invariant) extension $T_{U_0}M = E^s \oplus E^{cu}$ of the splitting $T_\Lambda M = E^s \oplus E^{cu}$. Given $x \in U_0$ and $a > 0$ we define the cone fields

$$\mathcal{C}^s_x(a) = \{v = v^s + v^{cu} \in E^s_x \oplus E^{cu}_x : \|v^{cu}\| \leq a\|v^s\|\},$$
$$\mathcal{C}^{cu}_x(a) = \{v = v^s + v^{cu} \in E^s_x \oplus E^{cu}_x : \|v^s\| \leq a\|v^{cu}\|\}.$$

Partial hyperbolic cone fields in U_0

Proposition
Fix T so that $\lambda^T = 1/150$. For any $a \in (0, \frac{1}{4}]$ there is a positively invariant neighborhood U_0 of Λ, s.t. $\forall x \in U_0$

(a) $DX^{-t}\big(\mathcal{C}^s_{X^t x}(b)\big) \subset \mathcal{C}^s_x(b)$ and $DX^t\big(\mathcal{C}^{cu}_x(b)\big) \subset \mathcal{C}^{cu}_{X^t x}(b)$, for all $b \geq a$, $t \geq T$ (*backward invariance of stable cones and forward invariance of center-unstable cones*).

(b) $\exists c > 0, \tilde{\lambda} \in (0,1)$ s.t. $\forall t > 0$

$$\|DX^{-t}(X^t x)v\| \geq c\tilde{\lambda}^{-t}\|v\|, \quad \forall v \in \mathcal{C}^s_{X^t x}(a);$$
$$\frac{\|DX^t(x)v\|}{\|v\|} \geq c\tilde{\lambda}^{-t}\frac{\|DX^t(x)u\|}{\|u\|} \quad \text{for} \begin{cases} \vec{0} \neq v \in \mathcal{C}^{cu}_x(a) \\ u \in DX^{-t}(\mathcal{C}^s_{X^t x}(a)) \end{cases};$$

(*backward expansion of stable cones and domination*).

(Skip the proof of this proposition)

Proof of the Proposition (extending cones)
If v lies in $T_x M$ where $x \in U_0$, then we write $v = v^s + v^{cu} \in E^s_x \oplus E^{cu}_x$. If $v \in \mathcal{C}^*_x(a)$, then $(1-a)\|v^*\| \leq \|v\| \leq (1+a)\|v^*\|$ where throughout $* \in \{s, cu\}$.

For $x \in \Lambda$, it follows from invariance of the splitting $E^s \oplus E^{cu}$ that $(DX_t(x)v)^* = DX_t(x)v^*$ for all $v \in T_x M$ and $t \in \mathbb{R}$.

We fix the ngbh. U_0 as follows. For each $x \in \Lambda$, we choose a ngbh. $U_x \subset M$ of x s.t. U_x is diffeomorphic to $\mathbb{R}^d$ where $d = \dim M$. Then $T_{U_x}M$ is identified with $U_x \times \mathbb{R}^d$. Given $y_1, y_2 \in U_x$, a vector $v \in \mathbb{R}^d$ corresponds to vectors $v_{y_j} \in T_{y_j}M$ via this identification.

By the smoothness of the flow, we can choose U_x so small that $\|DX_t(y_1)v_{y_1}\| \leq 2\|DX_t(y_2)v_{y_2}\|$ for all $x \in \Lambda$, $y_1, y_2 \in U_x$, $v \in \mathbb{R}^d$, $t \in [-T, T]$.

Fixing coordinate systems
By the continuity of the splitting $E^s \oplus E^{cu}$, for $a > 0$ fixed we can ensure for all $b \geq a/8$, $t \in [-T, T]$, that
if $DX_t(y_1)v_{y_1} \in \mathcal{C}^*_{y_1}(b)$**, then** $DX_t(y_2)v_{y_2} \in \mathcal{C}^*_{y_2}(2b)$**.**

We now fix U_0 to be a positively invariant neighborhood of Λ contained in $\bigcup_{x\in\Lambda} U_x$. By construction, for every $y \in U_0$, there exists $x \in \Lambda$ such that

(i) $DX_t(x)v_x \subset \mathcal{C}^*_x(b) \implies DX_t(y)v_y \subset \mathcal{C}^*_y(2b)$,

(ii) $DX_t(y)v_y \subset \mathcal{C}^*_y(b) \implies DX_t(x)v_x \subset \mathcal{C}^*_x(2b)$, and

(iii) $\frac{1}{2}\|DX_t(x)v_x\| \leq \|DX_t(y)v_y\| \leq 2\|DX_t(x)v_x\|$,

for all $v \in \mathbb{R}^d$, $b \geq a/8$, $t \in [-T, T]$.

Proof of item (a) of the proposition

From domination on the initial splitting over Λ we get

$$\begin{aligned}\|(DX_t(x)v)^s\| = \|DX_t(x)v^s\| &\le \|DX_t|E^s_x\|\|v^s\| \\ &\le \lambda^t\|DX_{-t}|E^{cu}_{X_tx}\|^{-1}\|v^s\| \\ &= \lambda^t\|(DX_t|E^{cu}_x)^{-1}\|^{-1}\|v^s\| \\ &\le \lambda^t\|(DX_t(x)v)^{cu}\|\|v^{cu}\|^{-1}\|v^s\|,\end{aligned}$$

for all $x\in\Lambda$, $v\in T_xM$, $t\ge 0$. In particular

$$DX_t(\mathcal{C}^{cu}_x(b))\subset\mathcal{C}^{cu}_{X_tx}(b\lambda^t),\quad \forall x\in\Lambda, b>0, t\ge 0.$$

From Λ to U_0

Now let $y\in U_0$, $b\ge a$, $v\in\mathcal{C}^{cu}_y(b)$. We can pass to a nearby point $x\in\Lambda$ with corresponding vector $v_x\in\mathcal{C}^{cu}_x(2b)$ by (ii). Then $DX_t(x)v_x\in\mathcal{C}^{cu}_{X_tx}(2b\lambda^t)$ for all $t\ge 0$. In particular, since $\lambda^T=1/150\le 1/16$,

$$DX_T(x)v_x\in\mathcal{C}^{cu}_{X_Tx}(b/8)\quad\text{and}\quad DX_t(x)v_x\in\mathcal{C}^{cu}_{X_tx}(2b),\ \forall t\ge 0.$$

From (i) we get

$$\begin{aligned}DX_T(\mathcal{C}^{cu}_y(b))&\subset\mathcal{C}^{cu}_{X_Ty}(b/4)\subset\mathcal{C}^{cu}_{X_Ty}(b)\quad\text{and}\\ DX_r(\mathcal{C}^{cu}_y(b))&\subset\mathcal{C}^{cu}_{X_ry}(4b),\quad \forall r\in[0,T], y\in U_0.\end{aligned}$$

By positive invariance of U_0, it follows inductively that $DX_{kT}(\mathcal{C}^{cu}_y(b))\subset\mathcal{C}^{cu}_{X_{kT}y}(b/4)\subset\mathcal{C}^{cu}_{X_{kT}y}(b)$ for all $y\in U_0$, $k\in\mathbb{Z}^+$.

The general $t\ge T$

For general $t\ge T$, write $t=kT+r$ where $k\ge 1$ and $r\in[0,T)$. Again using positive invariance of U_0 together with cone invariance

$$DX_t(\mathcal{C}^{cu}_y(b))=DX_{kT}\cdot DX_r(\mathcal{C}^{cu}_y(b))\subset DX_{kT}(\mathcal{C}^{cu}_{X_ry}(4b))\subset\mathcal{C}^{cu}_{X_ty}(b).$$

This completes the proof of forward invariance for the center-unstable cone fiels, and the proof of the backward invariance for the stable cone field is completely analogous.

Hence we have proved item (a) in the statement of the proposition.

Proof of item (b) of the proposition

Keep the choices of T and U_0 and recall that $a\in(0,\frac{1}{4}]$ is fixed. First we backward contraction along the stable cone field.

Suppose that $x \in \Lambda$ and $v \in \mathcal{C}^s_{X_T x}(2a)$. By backward invariance $DX_{-T}(X_T x)v \in \mathcal{C}^s_x(2a)$, so using the contraction on E^s_Λ

$$
\begin{aligned}
\|DX_{-T}(X_T x)v\| &\geq (1-2a)\|(DX_{-T}(X_T x)v)^s\| \\
&= (1-2a)\|(DX_T(x))^{-1}v^s\| \geq (1-2a)\lambda^{-T}\|v^s\| \\
&\geq (1+2a)^{-1}(1-2a)\lambda^{-T}\|v\| \\
&\geq 50\|v\| \geq 8\|v\|.
\end{aligned}
$$

Backward contraction from Λ to U_0

Now let $y \in U_0$, $v \in \mathcal{C}^s_{X_T y}(a)$. As in part (a), we can pass to a nearby point $x \in \Lambda$ with corresponding vector $v_x \in \mathcal{C}^s_{X_T x}(2a)$ and so $\|DX_{-T}(X_T x)v_x\| \geq 8\|v_x\|$. Using (iii) together with positive invariance of U_0, we have that $\|DX_{-T}(X_T y)v\| \geq 2\|v\|$ for all $v \in \mathcal{C}^s_{X_T y}(a)$.

By positive invariance of U_0 and backward invariance of the stable cone field, it follows inductively that

$$
\|DX_{-kT}(X_{kT} y)v\| \geq 2^k\|v\| \quad \text{for } y \in U_0,\, v \in \mathcal{C}^s_{X_{kT} y}(a),\, k \geq 0.
$$

For $t = kT + r$ where $k \in \mathbb{Z}^+$, $r \in [0,T)$, let $v \in \mathcal{C}^s_{X_t y}(a)$. Then $DX_{-t}(X_t y)v = DX_{-r}(X_r y)\, DX_{-kT}(X_t y)v$ so it follows from the previous estimates

$$
\|DX_{-t}(X_t y)v\| \geq c\|DX_{-kT}(X_{kT}(X_r y))v\| \geq c2^k\|v\|,
$$

where $c = \inf_{r\in[0,T], y\in U_0, v\in T_yM,\, v\neq 0} \|DX_{-r}(y)v\|/\|v\| > 0$. This completes the proof of backward contraction.

Proof of domination of the cone fields

From domination in Λ we get for $x \in \Lambda$, $u, v \in T_xM$,

$$
\frac{\|DX_T(x)u^s\|}{\|u^s\|} \leq \|DX_T|E^s_x\| \leq \lambda^T\|(DX_T|E^{cu}_x)^{-1}\|^{-1} \leq \lambda^T\frac{\|DX_T(x)v^{cu}\|}{\|v^{cu}\|}.
$$

Let $u \in DX_{-T}(\mathcal{C}^s_{X_T x}(2a))$, $v \in \mathcal{C}^{cu}_x(2a)$. By cone invariance

$$
\begin{aligned}
\frac{\|DX_T(x)v^{cu}\|}{\|v^{cu}\|} &\leq \frac{(1+2a)\|DX_T(x)v\|}{(1-2a)\|v\|}, \quad \text{and} \\
\frac{\|DX_T(x)u\|}{\|u\|} &\leq \frac{(1+2a)\|DX_T(x)u^s\|}{(1-2a)\|u^s\|},
\end{aligned}
$$

and so

$$
\frac{\|DX_T(x)u\|}{\|u\|} \leq 9\lambda^T\frac{\|DX_T(x)v\|}{\|v\|} \leq \frac{3}{50}\frac{\|DX_T(x)v\|}{\|v\|}
$$

for all $v \in \mathcal{C}^{cu}_x(2a)$, $u \in DX_{-T}(\mathcal{C}^s_{X_T x}(2a))$.

Again from Λ to U_0 and conclusion
Using (iii) it follows that

$$\frac{\|DX_T(y)u\|}{\|u\|} \leq \frac{24}{25} \frac{\|DX_T(y)v\|}{\|v\|}$$

for all $y \in U_0$, $v \in \mathcal{C}_y^{cu}(a)$, $u \in DX_{-T}(\mathcal{C}^s_{X_T y}(a))$.

For general $t \geq 0$, we write $t = kT + r$, $k \geq 0$, $r \in [0, T)$ and proceed as in the proof of item (a).

This completes the proof of the proposition on cone invariance, backward contraction on stable cones and domination for the cone fields in a neighborhood U_0 of the attracting set Λ.

2.2 Extended bundles

Invariant stable bundle extended to U_0
Whereas the original splitting $T_\Lambda M = E^s \oplus E^{cu}$ is DX^t-invariant, in general **the extension E^{cu} of the center-unstable direction cannot be assumed invariant**. However we have

Proposition
The continuous bundle E^s over U_0 can be chosen to be DX^t-invariant and uniformly contracting: $\|DX^t \mid E^s_x\| \leq c^{-1}\tilde{\lambda}^t$ for all $t \geq 0$, $x \in U_0$, where $c > 0$, $\tilde{\lambda} \in (0, 1)$ are the constants in the previous Proposition.

Impossible to extend the central bundle
Let us assume that the extension $E^{cu}_{U_0}$ is invariant.

Lemma
Let Λ be a compact invariant set for a flow X^t of a C^1 vector field X on M and assume Λ contains a Lorenz-like singularity σ. Given a continuous DX^t-invariant splitting $T_U M = E \oplus F$ on a neighborhood U of σ such that E is uniformly contracted, then there exists a ngbh. V of σ s.t $V \subset \overline{V} \subset U$ and a point $x_0 \in V \setminus \Lambda$ satisfying $X(x_0) \in F_x$.

However, for $x_0 \in V \setminus \Lambda$ close to the singularity, we have for some $t > 0$ that $x_s = X^s(x_0) \in U$ for all $-t < s < 0$, x_s is close to $W^{ss}(\sigma) \setminus \{\sigma\}$ and $G(x_{-t})$ is almost parallel to E^{ss}_σ.

This is a contradiction since the angle between E^{ss} and E^{cu} is bounded away from zero (see next picture).

Behaviour in small neighborhood of σ

(Skip the proof of the Lemma)

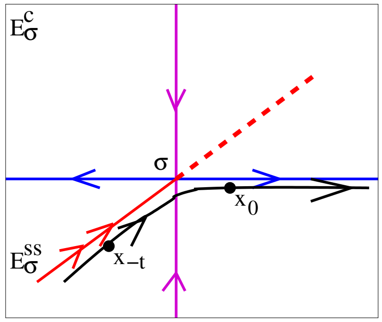


Figure 1: The flow direction contained in E^{cu} in a neighborhood of σ implies that E^{cu} is not continuous at σ.

Proof of the Lemma

We denote by $\pi(E_x) : T_xM \to E_x$ the projection on E_x parallel to F_x at T_xM, and likewise $\pi(F_x) : T_xM \to F_x$ is the projection on F_x parallel to E_x. We note that for $x \in U$

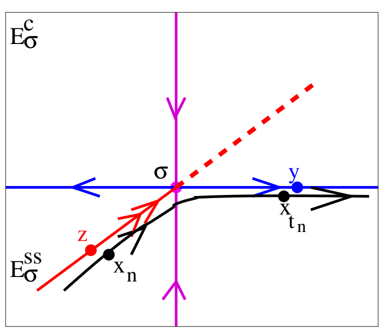
$$X(x) = \pi(E_x) \cdot X(x) + \pi(F_x) \cdot X(x)$$

and for $t > 0$ and $x \in V$ such that $X^{[0,t]}(x) \in U$, by linearity of DX^t and DX^t-invariance of the splitting $E \oplus F$

$$\begin{aligned} DX^t \cdot X(x) &= DX^t \cdot \pi(E_x) \cdot X(x) + DX^t \cdot \pi(F_x) \cdot X(x) \\ &= \pi(E_{X^t(x)}) \cdot DX^t \cdot X(x) + \pi(F_{X^t(x)}) \cdot DX^t \cdot X(x). \end{aligned}$$

Assuming that $\pi(E_x) \cdot X(x) \neq \vec{0}$ for all $x \in V \setminus \Lambda$, we choose a sequence of points $x_n \in V$ and of times $t_n > 0$ such that $t_n \nearrow \infty$ as follows.

Choice of the sequence of orbit segments in U



Let $x_n \in V$ be a sequence converging to $z \in W^{ss}_{loc}(\sigma) \setminus \{\sigma\}$ and $t_n \nearrow +\infty$ so that $X^{[0,t_n]}(x_n) \subset U$ and $x_{t_n} = X^{t_n}(x_n)$ tends to $y \in W^u_{loc}(\sigma) \setminus \{\sigma\}$.

Exploring the invariance and backward expansion

Since $\pi(E_x)\cdot X(x)\neq\vec{0}$ we get

$$\lim_{n\to+\infty} DX^{-t_n}\cdot X(x_{t_n}) = \lim_{n\to+\infty} X(x_n) = X(z) \quad \text{but also}$$
$$\|DX^{-t_n}\cdot\pi(E_{x_{t_n}})\cdot X(x_{t_n})\| \geq ce^{\tilde{\lambda}t_n}\|\pi(E_{x_{t_n}})\cdot X(x_{t_n})\| \xrightarrow[n\to+\infty]{} +\infty,$$

because $x_{t_n}\to y$ and E^s is a continuous bundle by assumption.

This is possible only if the angle between E_{x_n} and F_{x_n} tends to zero when $n\to+\infty$.

Closing angles

Indeed, using the Riemannian metric on T_xM, the angle $\alpha(x)=\alpha(E_x,F_x)$ between E_x and F_x is related to the norm of $\pi(E_x)$ as follows: $\|\pi(E_x)\| = 1/\sin(\alpha(x))$. Thus

$$\begin{aligned}\|DX^{-t_n}\cdot\pi(E_{x_{t_n}})\cdot X(x_{t_n})\| &= \|\pi(E_{x_n})\cdot DX^{-t_n}\cdot X(x_{t_n})\|\\ &\leq \frac{1}{\sin(\alpha(x_n))}\cdot\|X(x_n)\|, \quad \forall n\geq 1.\end{aligned}$$

Hence, because $\|X(x_n)\|\to\|X(z)\|\neq 0$ we deduce that $\alpha(x_n)\to 0$.

However, since $E\oplus F$ is a continuous splitting in U, then $E\oplus F$ are bounded away from zero in $\overline{V}$, which gives a contradiction.

We conclude that in V there must exist a point x_0 as in the statement of the lemma.

Proof of the Proposition (invariant E^s)

We begin with the original choice of continuous splitting $T_{U_0}M = E^s\oplus E^{cu}$. Let $a\in(0,\frac{1}{4}]$ and choose T and U_0 as in the Proposition on cone invariance and domination.

For $x\in U_0$, define (as usual in hyperbolic dynamics)

$$F_x = \bigcap_{t\geq 0} DX_{-t}\big(\mathcal{C}^s_{X_tx}(a)\big).$$

We show that $\{F_x\}$ is the desired stable bundle. That is, we show that for all $t\geq 0$,

(i) $x\mapsto F_x$ is a continuous map from U_0 to the Grassmannian bundle $\mathcal{G}=\{\mathcal{G}_x,\, x\in U_0\}$ where $\mathcal{G}_x$ is the space of d_s-dimensional subspaces of T_xM,

(ii) $F_x = E^s_x$ for $x\in\Lambda$,

(iii) $\{F_x,\, x\in U_0\}$ is DX_t-invariant and uniformly contracting.

(Skip the proof of the Proposition)

Nested family of cones and subspace contained in the intersection

Now $\{DX_{-t}(\mathcal{C}^s_{X_tx}(a)),\, t \ge 0\}$ is a nested family of closed cones, and by backward invariance, the cones are contained in $\mathcal{C}^s_x(a)$ for $t \ge T$. In particular, $F_x \subset \mathcal{C}^s_x(a)$.

We can also regard $\{DX_{-t}(\mathcal{C}^s_{X_tx}(a)),\, t \ge 0\}$ as a nested family of closed subsets of $\mathcal{G}_x$, so F_x is a closed subset of $\mathcal{G}_x$.

By compactness of $\mathcal{G}_x$, the elements $DX_{-t}E^s_{X_tx} \in \mathcal{G}_x$ have a convergent subsequence $DX_{-t_n}E^s_{X_{t_n}x}$ with limit $\tilde{F}_x \in \mathcal{G}_x$.

Since $DX_{-t}E^s_{X_tx} \in DX_{-t}(\mathcal{C}^s_{X_tx}(a))$ and F_x is closed, it follows that $\tilde{F}_x \in F_x$.

Uniqueness of the subspace in the intersection

To summarise, we have shown that there exists a d_s-dimensional subspace $\tilde{F}_x$ such that $\tilde{F}_x \subset F_x$ and $\tilde{F}_x = \lim_{n\to\infty} DX_{-t_n}E^s_{X_{t_n}x}$ (in $\mathcal{G}_x$). Without loss we may suppose that $t_n \ge T$ for all n.

Next we get $F_x = \tilde{F}_x$. Choose vectors $u_n \in E^s_{X_{t_n}x}$ s.t. $\|DX_{-t_n}(X_{t_n}x)u_n\| = 1$.

Suppose for contradiction that $F_x \ne \tilde{F}_x$. Then F_x is a nontrivial cone containing $\tilde{F}_x$, and so there exists $v \in E^{cu}_x$ nonzero such that $w_n = DX_{-t_n}(X_{t_n}x)u_n + v \in F_x$ for n sufficiently large. It follows from the definition of F_x that $DX_{t_n}(x)w_n = u_n + DX_{t_n}(x)v \in \mathcal{C}^s_{X_{t_n}x}(a)$. Hence

$$\|(DX_{t_n}(x)v)^{cu}\| \le a\|u_n + (DX_{t_n}(x)v)^s\|.$$

Uniqueness from domination

Since $v \in E^{cu}_x$, it follows from forward invariance that $DX_{t_n}(x)v \in \mathcal{C}^{cu}_x(a)$ and hence we obtain

$$\|(DX_{t_n}(x)v)^s\| \le a\|(DX_{t_n}(x)v)^{cu}\| \quad \text{and}$$
$$\|DX_{t_n}(x)v\| \le (1+a)\|(DX_{t_n}(x)v)^{cu}\|.$$

Substituting into the last inequality yields $(1-a^2)\|(DX_{t_n}(x)v)^{cu}\| \le a\|u_n\|$ and then

$$\|DX_{t_n}(x)v\| \le (1+a)(1-a^2)^{-1}a\|u_n\|.$$

On the other hand, $u_n \in E^s_{X_{t_n}x}$, $v \in E^{cu}_x$, so by domination

$$\frac{\|DX_{t_n}(x)v\|}{\|v\|} \ge c\tilde{\lambda}^{-t_n}\frac{\|u_n\|}{\|DX_{-t_n}(X_{t_n}x)u_n\|} = c\tilde{\lambda}^{-t_n}\|u_n\|.$$

Letting $n \to \infty$ yields the desired contradiction, and so F_x and $\tilde{F}_x$ coincide. In particular, $F_x \in \mathcal{G}_x$ for all $x \in U_0$.

Continuity of the family of subspaces

To prove continuity of the map $x \mapsto F_x$, fix $x \in U_0$ and let $\mathcal{U} \subset \mathcal{G}$ be a neighborhood of F_x.

There exists $t_0 \geq 0$ such that $\bigcap_{t \leq t_0} DX_{-t}(\mathcal{C}^s_{X_t x}(a)) \subset \mathcal{U}$.

By smoothness of the flow, $F_y \subset \bigcap_{t \leq t_0} DX_{-t}(\mathcal{C}^s_{X_t y}(a)) \subset \mathcal{U}$ for y sufficiently close to x.

This completes the proof of (i).

It is immediate from invariance of the bundle $E^s|_\Lambda$ that $E^s_x \subset F_x$ for all $x \in \Lambda$.

Since the dimensions are the same, $E^s_x = F_x$ **for all** $x \in \Lambda$ **establishing item (ii).**

Invariance and uniform contraction

For $r \geq 0$,

$$
\begin{aligned}
DX^r F_x &= \bigcap_{t \geq 0} DX^{r-t}(\mathcal{C}^s_{X^{t-r}(X^r x)}(a)) = \bigcap_{t \geq r} DX^{r-t}(\mathcal{C}^s_{X^{t-r}(X^r x)}(a)) \\
&= \bigcap_{t \geq 0} DX^{-t}(\mathcal{C}^s_{X^t(X^r x)}(a)) = F_{X^r x},
\end{aligned}
$$

so the bundle $\{F_x\}$ is DX^t-invariant.

Finally, if $v \in F_x$, $t \geq 0$, then $DX^t(x)v \in \mathcal{C}^s_{X^t x}(a)$ so by backward expansion on stable cones, $\|v\| \geq c\tilde{\lambda}^{-t}\|DX^t(x)v\|$.

Hence $\|DX^t \mid F_x\| \leq c^{-1}\tilde{\lambda}^t$ so item (iii) holds.

This completes de proof of the proposition on existence of invariant extension of the stable direction from Λ to a full neighborhood U_0 of Λ in the ambient space.

3 Stable Foliation

Stable foliation in a neighborhood of Λ

Existence of a flow invariant contracting stable manifold bundle W^s over U_0 consisting of C^1 leaves tangent to E^s.

From now on, we suppose that the continuous extension $T_{U_0}M = E^s \oplus E^{cu}$ of $T_\Lambda M = E^s \oplus E^{cu}$ is chosen so that E^s is invariant and uniformly contracted.

3.1 Existence

Existence of stable foliation in U_0

Let $\mathcal{D}^k$ denote the k-dimensional open unit disk and let $\mathrm{Emb}^r(\mathcal{D}^k, M)$ denote the set of C^r embeddings $\phi : \mathcal{D}^k \to M$ endowed with the C^r distance.

Theorem

There is a positively invariant neighborhood U_0 of Λ, and a constant $0 < \nu < 1$ s.t.

(a) $\forall x \in U_0 \exists W^s_x \in \mathrm{Emb}^r(\mathcal{D}^{d_s}, M)$ with $x \in W^s_x$ s.t.

   (a) $T_x W^s_x = E^s_x$.

   (b) $X^t(W^s_x) \subset W^s_{X^t x}, \forall t \geq 0$.

   (c) $d(X^t x, X^t y) \leq \nu^t d(x,y), \forall y \in W^s_x, t \geq 0$.

(b) there is a continuous map $\gamma : U_0 \to \mathrm{Emb}^0(\mathcal{D}^{d_s}, M)$ such that $\gamma(x)(0) = x$ and $\gamma(x)(\mathcal{D}^{d_s}) = W^s_x$.

(c) $\{W^s_x : x \in U_0\}$ defines a topological foliation of U_0.

(Skip the proof of the Theorem)

Proof of existence of stable foliation on U_0

We follow the exposition on Section 6.4(b) of the book by **Katok and Hasselblat, *Introduction to the Modern Theory of Dynamical Systems***, C.U.P., 1995.

Let $T > 0$, $c > 0$, $\tilde{\lambda} \in (0,1)$ be the constants in the propositions on existence of cone fields and extension of stable invariant directions to U_0.

Increase $T > 0$ if necessary so that $\hat{\lambda} = c^{-1}\tilde{\lambda}^T \in (0,1)$ and define the diffeomorphism $f = X_T : U_0 \to U_0$.

For each $x \in U_0$, we consider the exponential map $\exp_x : T_x M \to M$. This transforms a small enough neighborhood of 0 diffeomorphically onto a neighborhood of x, and $D\exp_x(0) = I$.

Setting of local adapted coordinates

Choose orthonormal bases on $\mathbb{R}^{d_s}$, $\mathbb{R}^{d_{cu}}$ and, for each $x \in U_0$, choose orthonormal bases on E^s_x and E^{cu}_x.

Let $P^s_x : \mathbb{R}^{d_s} \to E^s_x$, $P^{cu}_x : \mathbb{R}^{d_{cu}} \to E^{cu}_x$ be the corresponding isometric isomorphisms.

Since $U_0 \ni x \mapsto E^s_x \oplus E^{cu}_x$ is continuous, we can arrange that $x \mapsto P^s_x$ and $x \mapsto P^{cu}_x$ are continuous families of isomorphisms.

Define $P_{x,n} = P^s_{f^n x} + Df^n(x) P^{cu}_x : \mathbb{R}^d \to T_{f^n x} M$, which is a continuous family $x \mapsto P_{x,n}$ of isomorphisms for each n. In general $P_{x,n}$ is **not an isometric isomorphism**, since $Df^n \cdot E^{cu}_x$ is not necessarily orthogonal to $E^s_{f^n x}$.

However, we have $Df^n E_x^{cu} \subset \mathcal{C}^{cu}_{f^n x}(a)$ for some $a \in (0, \frac{1}{4}]$, so **the angle between the subspaces $E^s_{f^n x}$ and $Df^n E_x^{cu}$ is bounded away from zero.**

Hence there is a constant $C_1 \geq 1$ such that

$$\frac{1}{C_1} \leq \|P_{x,n}\| \leq C_1, \quad \forall x \in U_0, n \geq 0.$$

Next, $Q_{x,n} = \exp_{f^n x} \circ P_{x,n} : \mathbb{R}^d \to M$ maps a neighborhood of 0 in $\mathbb{R}^d$ diffeomorphically onto a neighborhood of $f^n x$ and $U_0 \ni x \mapsto Q_{x,n}$ is a continuous family of diffeomorphisms for each n.

Let $D_\rho \subset \mathbb{R}^d$ denote the ρ-neighborhood of 0. Using boundedness of $\|P_n\|$ and compactness of Λ, and shrinking U_0 if necessary, we can choose $\rho > 0$ so that $Q_{x,n} : D_\rho \to M$ is a diffeomorphism onto its range for all n. Moreover, there is a constant $C_2 \geq 1$ such that

$$C_2^{-1}\|p\| \leq d(f^n x, Q_{x,n}(p)) \leq C_2\|p\|,$$

for all $x \in U_0$, $n \geq 0$, $p \in D_\rho$.

Local expression for the dynamics

Now define the family $f_{x,n} = Q^{-1}_{x,n+1} \circ f \circ Q_{x,n} : D_\rho \to \mathbb{R}^d$.

By construction, $Df_{x,n}(0)$ is identified with $Df(f^n x)$ and $f_{x,n}$ are uniformly C^r close to $Df_{x,n}(0)$ on D_ρ.

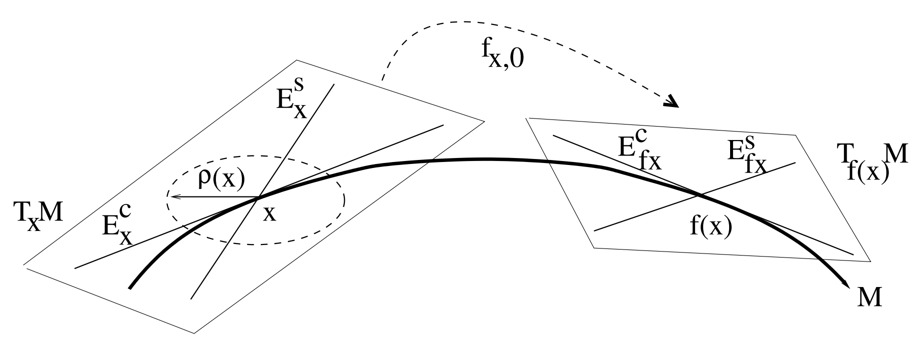
Hence for any $\delta > 0$ there exists $\rho > 0$ and a family of (surjective) C^r diffeomorphisms $g_{x,n} : \mathbb{R}^d \to \mathbb{R}^d$, $n \geq 0$, s.t. $\|g_{x,n} - Df_{x,n}(0)\|_{C^1} < \delta$ and $g_{x,n} = f_{x,n}$ on D_ρ. [For a proof of this standard result see e.g. Lemma 6.2.7 in Katok-Hasselblatt book cited above]

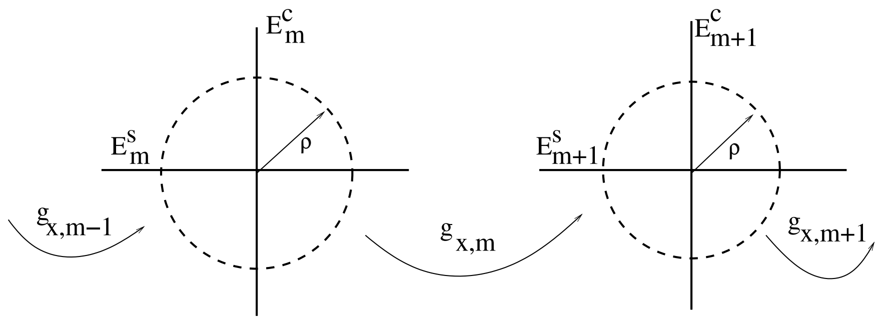
Proposition

For all $n \geq 0$ we have $\|Dg_{x,n}(0) \mid \mathbb{R}^{d_s}\| \leq \hat{\lambda}$ and

$$\|Dg_{x,n}(0) \mid \mathbb{R}^{d_s}\| \cdot \|Dg_{x,n}(0)^{-1} \mid \mathbb{R}^{d_{cu}}\| \leq \hat{\lambda}.$$

Dynamics in local coordinates





Dynamics in adapted coordinates

Proof of the proposition
Choose a as in the previous Proposition ensuring the existence of invariant cone fields in U_0.

By construction, $Dg_{x,n}(0) = Df_{x,n}(0)$ is identified with $Df(f^n x)$ and

$$\begin{aligned}\|Dg_{x,n}(0) \mid \mathbb{R}^{d_s}\| &= \|Df \mid E^s_{f^n x}\| = \|DX_T \mid DX_{-T}E^s_{X_T f^n x}\|,\\ \|Dg_{x,n}(0)^{-1} \mid \mathbb{R}^{d_{cu}}\| &= \|Df^{-1} \mid Df^{n+1}E^{cu}_x\|\\ &\le \|DX_{-T} \mid DX_T(\mathcal{C}^{cu}_{f^n x}(a))\|,\end{aligned}$$

where we have used invariance of E^s and forward invariance of $\mathcal{C}^{cu}(a)$.

The first estimate is immediate from the proposition on existence and contraction of the extension of the stable direction to U_0.

The second estimate follows from the domination on the cone fields, and concludes the proof.

A modified Invariant Manifold Theorem
We require a slightly modified version of the Hadamard-Perron Invariant Manifold Theorem from Theorem 6.2.8, pp 242-257 in Katok-Hasselblatt book.

The only difference from the proof of Theorem 6.2.8 in Katok-Hasselblatt is that the rates λ_n, μ_n may depend on n.

However, the ratios λ_n/μ_n are controlled uniformly, and it is easy to check that the proof in pp 242-257 of Katok-Hasselblatt is valid in this slightly more general setting with no change in the arguments.

We now state this result for future use.

A Hadamard-Perron Invariant Manifold Theorem
Fix $r \ge 1$, $\lambda_{min} > 0$ and $\sigma \in (0,1)$. Then there exists $\gamma, \delta > 0$ arbitrarily small so that: for each n let $g_n : \mathbb{R}^d \to \mathbb{R}^d$ be a C^r diffeo s.t.

$$g_n(u,v) = (A_n u + \alpha_n(u,v), B_n v + \beta_n(u,v)), \quad (u,v) \in \mathbb{R}^{d_s} \oplus \mathbb{R}^{d_{cu}},$$

for linear maps $A_n : \mathbb{R}^{d_s} \to \mathbb{R}^{d_s}, B_n : \mathbb{R}^{d_{cu}} \to \mathbb{R}^{d_{cu}}$ and C^r maps $\alpha_n : \mathbb{R}^d \to \mathbb{R}^{d_s}, \beta_n : \mathbb{R}^d \to \mathbb{R}^{d_{cu}}$ with

$$\alpha_n(0,0) = 0, \beta_m(0,0) = 0 \quad \text{and} \quad \|\alpha_n\|_{C^1} < \delta, \|\beta_n\|_{C^1} < \delta.$$

Define $\lambda_n = \|A_n\|$, $\mu_n = \|B_n^{-1}\|^{-1}$ and suppose that $\lambda_n \geq \lambda_{min}$ and $\lambda_n/\mu_n \leq \sigma$.

Set $\lambda'_n = (1+\gamma)(\lambda_n + \delta(1+\gamma))$, $\mu'_n = \frac{\mu_n}{1+\gamma} - \delta$ and suppose that $\lambda'_n < \nu_n < \mu'_n$ for all $n \in \mathbb{Z}$.

Then **there exists a unique family of d_s-dimensional C^1 manifolds**

$$Z_n = \{(x, \varphi_n(x)) : x \in \mathbb{R}^{d_s}\},$$

where $\varphi_n : \mathbb{R}^{d_s} \to \mathbb{R}^{d_{cu}}$ satisfies for all $n \in \mathbb{Z}$

$$\varphi_n(0,0) = 0, \quad D\varphi_n(0,0) = 0 \quad \text{and} \quad \|D\varphi_n\|_{C^0} < \gamma,$$

and the following properties hold

1. $g_n(Z_n) = Z_{n+1}$,
2. $\|g_n(q)\| \leq \lambda'_n \|q\|$ for $q \in Z_n$,
3. If $\|g_{n+k-1} \circ \cdots \circ g_n(q)\| \leq C\nu_{n+k-1} \ldots \nu_n \|q\|$ for all $k \geq 0$ and some $C > 0$, then $q \in Z_n$.

If $\sup_n \lambda_n < 1$ (i.e. we have uniform contraction), then the manifolds Z_n are C^r.

Verifying the conditions of the theorem

Fix $x \in U_0$. The sequence of diffeos $g_{x,n} : \mathbb{R}^d \to \mathbb{R}^d$ is defined for $n \geq 0$.

For $n < 0$, we set $g_{x,n} = g_{x,0}$. The diffeos $g_{x,n}$ now have the structure required in the theorem.

Take $\sigma = \hat{\lambda} \in (0,1)$ and $\lambda_{min} = \inf_{x \in U_0} \|DX_T \mid E^s_x\| > 0$. By Proposition on adapted coordinates, the linear maps A_n, B_n satisfy the constraints $\lambda_{min} \leq \lambda_n \leq \sigma$ and $\lambda_n/\mu_n \leq \sigma$.

Choose $\gamma, \delta > 0$ so small that $\sup_n \lambda'_n < 1$ and $\sup_n \lambda'_n/\mu'_n < 1$.

Choose $\nu_n \in (\lambda'_n, \mu'_n)$ such that $\nu = \sup_n \nu_n < 1$. Finally, shrink ρ so that $\|\alpha_n\|_{C^1} < \delta$, $\|\beta_n\|_{C^1} < \delta$.

This shows that the hypotheses of the theorem are satisfied, with $\nu_n \leq \nu < 1$ for all n.

Using the conclusion of the theorem

Let $Z_{x,n}$ denote the family of d_s-dimensional C^r manifolds and set $W^s_x = Q_{x,0}(Z_{x,0} \cap D_\rho)$.

Repeating the construction for every $x \in U_0$, we get a family $\mathcal{F}^{ss} = \{W^s_x,\, x \in U_0\}$ of d_s-dimensional C^r manifolds covering U_0.

Lemma ($\mathcal{F}^{ss}$ is the desired family of stable manifolds)
Let $x, y \in U_0$. Then for all $n \geq 0$

(a) $d(x,y) < C_2^{-1}\rho, y \in W_x^s \implies d(f^n x, f^n y) \le C_2^2 \nu^n d(x,y)$.

(b) Let $C > 0$. If $d(x,y) < C_2^{-1}C^{-1}\rho$ and $d(f^n x, f^n y) \le C\nu^n d(x,y)$ for all $n \ge 0$, then $y \in W_x^s$.

(c) There exists $\varepsilon > 0$ such that if $d(x,y) < \varepsilon$ and $y \in W_x^s$ then $fy \subset W_{fx}^s$.

(Skip the proof of the Lemma)

Proof of the lemma

Let $F_{x,n} = f_{x,n-1} \circ \cdots \circ f_{x,0}$, $\quad G_{x,n} = g_{x,n-1} \circ \cdots \circ g_{x,0}$. Note that if $F_{x,n}(q) \in D_\rho$ for all $0 \le n \le N_0$, or if $G_{x,n}(q) \in D_\rho$ for all $0 \le n \le N_0$, then $F_{x,n}(q) = G_{x,n}(q)$ for all $0 \le n \le N_0$.

(a) Let $y \in W_x^s$ with $d(x,y) < C_2^{-1}\rho$. Then $q = Q_{x,0}^{-1}(y) \in Z_{x,0}$, so by (1-2) of the Inv. Manifold Thm.

$$\|G_{x,n}(q)\| \le \nu^n \|q\| = \nu^n \|Q_{x,0}^{-1}(y)\| \le \nu^n C_2 d(x,y) < \rho,$$

for all $n \ge 0$. Now $f^n = Q_{x,n} \circ F_{x,n} \circ Q_{x,0}^{-1}$, so

$$f^n y = Q_{x,n} \circ F_{x,n}(q) = Q_{x,n} \circ G_{x,n}(q).$$

Hence

$$d(f^n x, f^n y) = d(f^n x, Q_{x,n} \circ G_{x,n}(q)) \le C_2 \|G_{x,n}(q)\| \le C_2^2 \nu^n d(x,y)$$

completing the proof of item (a).

Characterizing the stable manifold

(b) Suppose that $d(x,y) < C_2^{-1}C^{-1}\rho$ and

$$d(f^n x, f^n y) \le C\nu^n d(x,y), \quad \forall n \ge 0.$$

Let $q = Q_{x,0}^{-1}(y)$ so $d(x,y) \le C_2 \|q\|$.

Now $F_{x,n} = Q_{x,n}^{-1} \circ f^n \circ Q_{x,0}$, so

$$\|F_{x,n}(q)\| = \|Q_{x,n}^{-1} \circ f^n(y)\| \le C_2 d(f^n x, f^n y) \le C_2 C \nu^n d(x,y) < \rho.$$

Hence

$$\|G_{x,n}(q)\| = \|F_{x,n}(q)\| \le C_2 C \nu^n d(x,y) \le C_2^2 C \nu^n \|q\|.$$

By item (3) of the Inv. Manif. Thm. $q \in Z_{x,0} \cap D_\rho$ and so $y = Q_{x,0}(q) \subset W_x^s$.

This completes the proof of item (b).

Forward invariance of the stable manifolds

(c) Let $x' = fx$, $y' = fy$ and choose $E \geq 1$ such that $d(x, y) \leq Ed(x', y')$ for all $x, y \in U_0$.

Suppose that $y \in W^s_x$ and $d(x, y) < C_2^{-5}E^{-1}\rho$. Then certainly, $d(x, y) < C_2^{-1}\rho$, so by part (a),

$$d(f^n x', f^n y') = d(f^{n+1}x, f^{n+1}y) \leq C_2^2 \nu^{n+1} d(x, y) \leq C_2^2 E \nu^n d(x', y') = C\nu^n d(x', y'),$$

where $C = C_2^2 E$.

Also, $d(x', y') \leq C_2^2 d(x, y) < C_2^{-3}E^{-1}\rho = C_2^{-1}C^{-1}\rho$, so the result follows from part (b).

This completes the proof of item (c) and of the lemma.

3.2 Topological foliation

The C^r embedded disks W^s_x depend continuously on x in the C^0 topology

Lemma

There is a continuous map $\gamma : U_0 \to \mathrm{Emb}^0(\mathcal{D}^{d_s}, M)$ such that $\gamma(x)(0) = x$ and $\gamma(x)(\mathcal{D}^{d_s}) = W^s_x$. Moreover, there exists $L \geq 1$ such that $\mathrm{Lip}\,\gamma(x) \leq L$ for all $x \in U_0$, where

$$\mathrm{Lip}\,\gamma(x) = \sup_{u \neq u'} \frac{d\big(\gamma(x)(u), \gamma(x)(u')\big)}{\|u - u'\|}.$$

(Skip the proof of the Lemma)

Proof of the continuity lemma

Fix $x \in U_0$ and recall that $W^s_x = Q_{x,0}(Z_{x,0} \cap D_\rho)$.

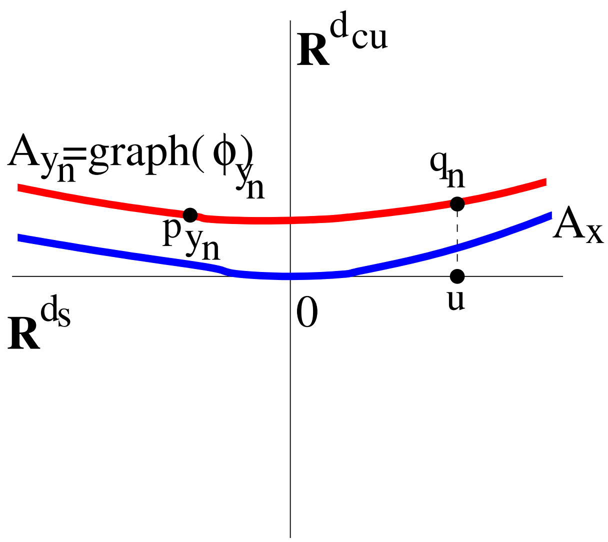
For y close to x, let $A_y = Q_{x,0}^{-1}(W^s_y)$. Let $p_y = Q_{x,0}^{-1}(y) = Q_{x,0}^{-1} \circ Q_{y,0}(0) \in A_y$.

In particular $A_x = Z_{x,0} \cap D_\rho$ and $p_x = 0$. Moreover, $y \mapsto p_y$ is continuous.

Now $T_{p_y}A_y = DQ_{x,0}^{-1}(y)T_y W^s_y = DQ_{x,0}^{-1}(y)E^s_y$, so it follows from smoothness of $Q_{x,0}$ and continuity of E^s that A_y can be viewed as a graph over $\mathcal{D}^{d_s} \subset \mathcal{R}^{d_s}$ for y close to x.

In particular, $A_y = \{(u, \phi_y(u)) : u \in \mathcal{D}^{d_s}\}$ where $\phi_y : \mathcal{D}^{d_s} \to \mathcal{R}^{d_{cu}}$.

Hence $W^s_y = \{Q_{x,0}(u, \phi_y(u)) : u \in \mathcal{D}^{d_s}\}$. The family of functions ϕ_y are C^r with uniform Lipschitz constant. Since $p_y \in A_y$, there exists $u_y \in \mathcal{D}^{d_s}$ such that $p_y = (u_y, \phi_y(u_y))$.



A_{y_n} as graph of ϕ_{y_n} near A_x.

Define the family of embeddings $\gamma : U_0 \to \mathrm{Emb}^r(\mathcal{D}^{d^s}, M)$ given by

$$\gamma(y)(u) = Q_{x,0}(u, \phi_y(u)).$$

We claim that $y \mapsto \phi_y$ is continuous at x in the C^0 topology, and hence the embedding γ is continuous at x in the C^0 topology.

Indeed, suppose that $y_n \to x$. By Arzelà-Ascoli, we can pass to a further subsequence such that $\lim_{n\to\infty} \sup_{u\in\mathcal{D}^{d_s}} \|\phi_{y_n}(u) - \psi(u)\| = 0$ for some continuous function $\psi : \mathbb{R}^{d_s} \to \mathbb{R}^{d_{cu}}$.

Since $p_{y_n} \to 0$, for n large enough we have that $p_{y_n} \in D_{\frac{1}{2}C_2^{-5}\rho}$.

Now fix $u \in \mathcal{D}^{d_s}$. Shrinking the disk $\mathcal{D}^{d_s}$, we can ensure that $q_n = (u, \phi_{y_n}(u)) \in D_{\frac{1}{2}C_2^{-5}\rho}$ for n sufficiently large. Hence

$$d(Q_{x,0}(q_n), y_n) \leq d(Q_{x,0}(q_n), x) + d(x, y_n) \leq C_2^{-3}\rho \leq C_2^{-1}\rho.$$

By construction, $Q_{x,0}(q_n) \in W^s_{y_n}$, so by item (a) of the existence lemma for the stable leaves

$$d(f^k \circ Q_{x,0}(q_n), f^k y_n) \leq C_2^2 \nu^k d(Q_{x,0}(q_n), y_n) \quad \text{for all } k \geq 0.$$

Letting $n \to \infty$, we obtain that

$$d(f^k \circ Q_{x,0}(u, \psi(u)), f^k x) \leq C_2^2 \nu^k d(Q_{x,0}(u, \psi(u)), x) \quad \text{for all } k \geq 0.$$

By item (b) of the existence lemma for the stable leaves $Q_{x,0}(u, \psi(u)) \in W^s_x$ so $(u, \psi(u)) \in A_x$. It follows that $\psi(u) = \phi_x(u)$.

Hence all subsequential limits of ϕ_y (as $y \to x$) coincide with ϕ_x so $\lim_{y \to x} \phi_y = \phi_x$ in the C^0 topology as required.

This completes the proof of the continuity of the stable manifolds with respect to the base point.

The stable manifolds are a topological foliation

Lemma
The family of disks $\{W^s_x : x \in U_0\}$ defines a topological foliation.

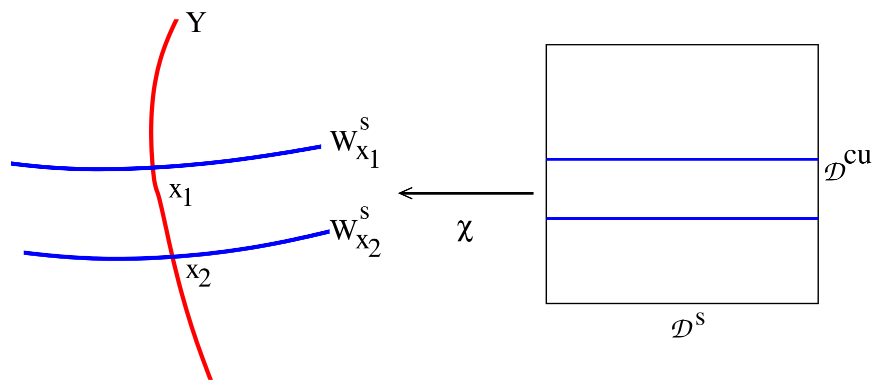
To prove this, let $x \in U_0$ and choose an embedded d^{cu}-dimensional disk $Y \subset M$ containing x and transverse to W^s_x.

By continuity of E^s, we can shrink Y so that Y is transverse to W^s_y at y for all $y \in Y$. Let $\psi : \mathcal{D}^{cu} \to Y$ be a choice of embedding and define $\chi : \mathcal{D}^s \times \mathcal{D}^{cu} \to U_0$ by setting

$$\chi(u, v) = \gamma(\psi(v))(u).$$

Note that χ maps horizontal lines $\{v = \text{const.}\}$ homeomorphically onto stable disks.

Topological foliation chart



By the previous lemma (on continuity of $U_0 \ni x \mapsto W^s_x$), each of these embeddings is Lipschitz with uniform Lipschitz constant L and using this together with continuity $d(\chi(u, v), \chi(u_0, v_0)) \leq$

$$\begin{aligned} &\leq d(\gamma(\psi(v))(u), \gamma(\psi(v))(u_0)) + d(\gamma(\psi(v))(u_0), \gamma(\psi(v_0))(u_0)) \\ &\leq L\|u - u_0\| + \|\gamma(\psi(v)) - \gamma(\psi(v_0))\|_{C^0} \to 0, \end{aligned}$$

as $(u, v) \to (u_0, v_0)$, establishing continuity of χ.

Suppose that $\chi(u_1, v_1) = \chi(u_2, v_2)$ with common value $y \in U_0$. Then $y \in W^s_{x_1} \cap W^s_{x_2}$ where $x_j = \psi(v_j)$.

We claim that $x_1 = x_2$ with common value $\hat{x}$. In particular $v_1 = v_2$.

But now $\gamma(\hat{x})(u_1) = \gamma(\hat{x})(u_2)$ and so $u_1 = u_2$. It follows that χ is injective and hence is a homeomorphism onto a neighborhood of x as required for $\{W^s_x\}_{x\in U_0}$ to be a topological foliation.

It remains to prove the claim.

Note that $W^s_{x_2}$ can be viewed as a graph over $W^s_{x_1}$. Let $A = W^s_{x_1} \cap W^s_{x_2}$. We show that A is open and closed in $W^s_{x_1}$. Since $y \in A$ and $W^s_{x_1}$ is connected, $A = W^s_{x_1}$ and in particular, $x_2 = x_1$ as required.

It is clear that A is closed in $W^s_{x_1}$. To prove that A is open, suppose that $z \in A$. Since $W^s_{x_j}$ are tangent to $E^s_{x_j}$ with uniform Lipschitz constant, there exists $C > 0$ such that $d(x_1, x_2) \le Cd(z, x_j)$ for $j = 1, 2$.

Let $z' \in W^s_{x_1}$ be such that $d(z, z') \le (1/2C)d(x_1, x_2)$.

Note that this implies $d(x_1, x_2) \le 2Cd(z', x_2)$.

We must show that $z' \in A$.

Now

$$
\begin{aligned}
&d(f^n z', f^n x_2) \le d(f^n z', f^n x_1) + d(f^n x_1, f^n z) + d(f^n z, f^n x_2) \\
&\le C_2^2 \nu^n \{d(z', x_1) + d(x_1, z) + d(z, x_2)\} \\
&\le C_2^2 \nu^n \{d(z', x_2) + d(x_2, x_1) \\
&\qquad + \; d(x_1, x_2) + d(x_2, z') + d(z', z) \; + \; d(z, z') + d(z', x_2)\} \\
&= C_2^2 \nu^n \{3d(z', x_2) + 2d(x_1, x_2) + 2d(z, z')\} \\
&\le C_2^2 \nu^n \{3d(z', x_2) + 4d(x_1, x_2)\} \\
&\le (3 + 8C) C_2^2 \nu^n d(z', x_2).
\end{aligned}
$$

We can arrange that χ takes values in $B_\varepsilon(x)$ where ε is as small as required.

By item (b) of the lemma on existence of stable manifolds, $z' \in W^s(x_2)$ and hence $z' \in A$ completing the proof.

Flow invariance of the foliation

Corollary
There exists $\varepsilon > 0$ such that $X^t(W^s_x \cap B_\varepsilon(x)) \subset W^s_{X^t x}$ for all $t \ge 0$, $x \in U_0$.

To prove this, choose $n_0 \ge 1$ such that $C_2^2 \nu^{n_0} < 1$.

Shrinking ε, it follows from items (a)-(c) of the lemma on existence of stable leaves, that $f^{n_0}(W^s_x \cap B_\varepsilon(x)) \subset W^s_{f^{n_0}x} \cap B_\varepsilon(f^{n_0}x)$ and, inductively, that $f^{kn_0}(W^s_x \cap B_\varepsilon(x)) \subset W^s_{f^{kn_0}x} \cap B_\varepsilon(f^{kn_0}x)$ for all $k \ge 0$.

Next choose $C \ge 1$ such that $d(X^r x, X^r y) \le Cd(x, y)$ for all $x, y \in U_0$, $r \in [-n_0 T, n_0 T]$.

Suppose that $y \in W^s_x$ and let $x' = X^r x$, $y' = X^r y$. By item (a) of the lemma on existence of stable leaves, for y sufficiently close to x and for all $n \ge 0$

$$
\begin{aligned}
d(f^n x', f^n y') &= d(X^r f^n x, X^r f^n y) \le Cd(f^n x, f^n y) \\
&\le CC_2^2 \nu^n d(x, y) \le C^2 C_2^2 \nu^n d(x', y').
\end{aligned}
$$

By item (b) of the same lemma, $X^r y \in W^s_{X^r x}$ for y sufficiently close to x.

Hence there exists $\varepsilon > 0$ such that $X^r(W^s_x \cap B_\varepsilon(x)) \subset W^s_{X^r x}$ for all $r \in [0, n_0 T]$, $x \in U_0$.

The result for general t follows by writing $t = kn_0T + r$ where $k \geq 0$, $r \in [0, n_0 T)$.

The proof is complete.

Completing the proof of existence of the stable foliation

Recall that $f = X^T$. Choose C such that $\sup_{r \in [0,T]} d(X^r x, X^r y) \leq C d(x, y)$ for all $x, y \in U$. Write $t = nT + r$, $n \geq 0$, $r \in [0, T)$.

By item (a) of the lemma on the existence of stable leaves, if $d(x, y) < C_2^{-1}\rho$ and $y \in W^s_x$, then

$$d(X^t x, X^t y) = d(X^{nT+r}x, X^{nT+r}y) \leq C_2^2 C \nu^n d(x, y) \leq C' \tilde{\nu}^t d(x, y),$$

where $C' = C_2^2 C \nu^{-1}$ and $\tilde{\nu} = \nu^{1/T}$.

Passing to an adapted metric, we can arrange that there are constants $\varepsilon > 0$, $\nu \in (0, 1)$ such that if $d(x, y) < \varepsilon$ and $y \in W^s_x$, then $d(X^t x, X^t y) \leq \nu^t d(x, y)$ for all $t \geq 0$.

From now on, we write W^s_x instead of $W^s_x \cap B_\varepsilon(x)$. With this notation, the previous Corollary states that $X^t(W^s_x) \subset W^s_{X^t x}$ for all $x \in U_0$, $t \geq 0$.

This completes the proof of the Theorem on the existence of a foliation everywhere tangent to the extension $\{E^s_x\}_{x \in U_0}$ of the stable bundle to the whole of U_0.

3.3 Smooth Foliation: bunching condition

Regularity of the stable foliation: with bunching

We recall that X^t is the flow generated by a C^r vector field G where $r \geq 2$. Let $q \in [0, r]$.

We suppose that there exists $t > 0$ so that the following bunching condition holds:

$$\|DX^t \mid E^s_x\| \cdot \|DX^{-t} \mid E^{cu}_{X^t x}\| \cdot \|DX^t \mid E^{cu}_x\|^q < 1 \quad \text{for all } x \in \Lambda.$$

Theorem

Let $q \in [0, [r]\,]$. If the q-bunching condition holds for some $t > 0$, then the bundle E^s is C^q over U_0. That is, the map $x \mapsto E^s_x$ is a C^q map from a smaller neighborhood $U_1 \subset U_0$ of Λ to $\mathcal{G}_1$ (the Grassmann of all one-dimensional subspaces on $T_{U_1}M$).

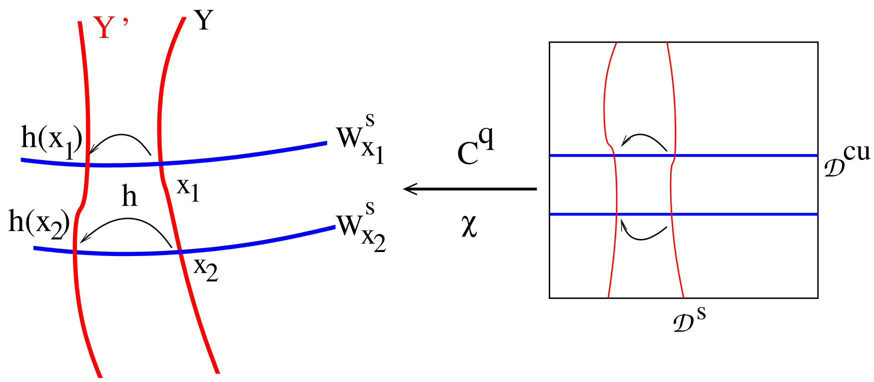
Consequences of smoothness of the stable bundle

1. It is immediate from domination that a q-bunching condition holds with $q = 0$. By smoothness of the flow and compactness of Λ, a q-bunching condition holds for some $q > 0$. **Hence the stable bundle E^s is at least Hölder over U_1.**

2. **When** $q \geq 1$ in the previous theorem, it follows by a theorem of Frobenius that **the family of stable manifolds** $\{W^s_x\}_{x \in U_0}$ already obtained **forms a** C^q **foliation of U_1, in the sense that the foliation charts are C^q.**

   Moreover, the holonomy maps along the stable leaves are C^q smooth.

Holonomies



Example of non-smooth bundle and holonomy

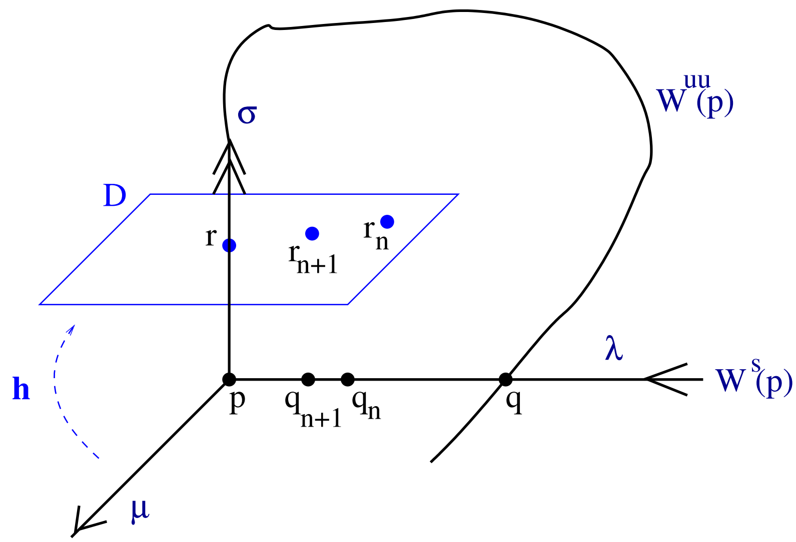
Let p be a fixed point of an Anosov diffeomorphism $f : \mathbb{T}^3 \to \mathbb{T}^3$ with the splitting $T_p\mathbb{T}^3 = E^s \oplus E^u \oplus E^{uu}$ into $1d$ non-trivial subspaces. We assume that f is locally smooth linearizable at a neighborhood U of p and (fixing an orientation)

$$0 < \lambda = \|Df_p \mid E^s\| \quad < 1 < \quad \mu = \|Df_p \mid E^u\| \quad < \quad \sigma = \|Df_p \mid E^u\|.$$

We also assume that there exists $q = (1, 0, 0) \in W^{uu}(p) \pitchfork W^s(p) \setminus \{p\}$ **in** U **such that** $T_q W^u(p) \ni v = (v^s, v^c, v^u)$ **with** $(v^c, v^u) \neq (0, 0)$**.**

We set $q_n = f^n q = (\lambda^n, 0, 0)$, $v_n = Df^n_q \cdot v = (\lambda^n v^s, \mu^n v^c, \sigma^n v^u)$ and, for a cross-section $D = \{z = 1\} \cap U$ in linearized coordinates, we set $r_n = hq_n$ and $r = hp$, where $n \geq 1$ and $h : \{z = 0\} \cap U \to D$ is the holonomy along the leaves of the strong-unstable foliation, tangent to the subbundle E^u.

Example of unsmooth foliation/holonomy



Smooth holonomy leads to a contradiction

If E^u is C^1, then h is C^1, thus

$$hq_n - hp = Dh_p \cdot (q_n - q) + L(p, q_n) \quad \text{with} \quad \frac{\|L(p,q_n)\|}{\|q_n - p_n\|} \xrightarrow[n\to\infty]{} 0$$

and so $\lim_n \frac{\|hq_n - hp\|}{\|q_n - p\|} = \|Dh_p \cdot e_1\| \neq 0$. However, in the linearized, if we write $hq_m = r_m = (r^s_m, r^c_m, 1)$ for some $m \geq 1$, then

$$hq_{n+m} = r_{m+n} = (\lambda^n r^s_m, \mu^n r^c_m, 1) \quad \text{with} \quad r^c_m \neq 0, n \geq 1.$$

Since $hp = r = (0,0,1)$ and $p = (0,0,0)$, we deduce that **if E^u (and so h) is smooth, then μ^n is comparable to λ^n.**

This contradiction shows that, in this example, the bundle E^u cannot be smooth.

(Skip the proof of the theorem)

Proof of the theorem

Choose t as in the q-bunching condition and set $f = X^t$.

Increasing t if necessary, we can ensure that

$$\|Df \mid E^s_x\| \|Df^{-1} \mid E^{cu}_{fx}\| \leq \|Df \mid E^s_x\| \cdot \|Df^{-1} \mid E^{cu}_{fx}\| \cdot \|Df \mid T_x M\|^q < 1,$$

for all $x \in U_0$. Let $T_{U_0} M = E^s \oplus E^{cu}$ be the continuous splitting with E^s invariant already constructed.

Take $T_{U_0} M = F^s \oplus F^{cu}$ a C^r approximation of this splitting and for each $x \in U_0$, let $L(F^s_x, F^{cu}_x)$ denote the space of linear maps from F^s_x to F^{cu}_x, and let $\mathbb{D}_x$ denote the unit disk in $L(F^s_x, F^{cu}_x)$ (with the norm induced by the Riemannian metric).

Define the corresponding disk bundle $\mathcal{D}_0 = \{\mathbb{D}_x, \, x \in U_0\}$.

Invariant section over overflowing diffeomorphism

Let $U_1 = f(U_0) \subset U_0$ and set $\mathcal{D}_1 = \{\mathbb{D}_x,\, x \in U_1\}$.

Let $h = f^{-1}|_{U_1} : U_1 \to U_0$. Since $h(U_1) = U_0 \supset U_1$, the C^r diffeomorphism h is **overflowing** in the sense of Hirsch-Pugh-Shub, *Invariant Manifolds*, '77.

Represent $Dh(x) : T_xM \to T_{hx}M$ using the splitting $F^s \oplus F^{cu}$ by writing

$$Dh(x) = \begin{pmatrix} A_x & B_x \\ C_x & D_x \end{pmatrix} : F^s_x \times F^{cu}_x \to F^s_{hx} \times F^{cu}_{hx}, \quad x \in U_1.$$

We define the graph transform $\Gamma : \mathcal{D}_1 \to \mathcal{D}_0$,

$$\Gamma_x(\ell) = (C_x + D_x\ell)(A_x + B_x\ell)^{-1}, \quad \ell \in \mathcal{D}_x,\, x \in U_1.$$

A Lemma and the Theorem

Lemma

The neighborhood U_0 of Λ and the C^r splitting $F^s \oplus F^{cu}$ can be chosen so that $\Gamma : \mathcal{D}_1 \to \mathcal{D}_0$ is well-defined and $\mathrm{Lip}(\Gamma_x) \cdot \|Dh^{-1}|T_{hx}M\|^q < 1$ for all $x \in U_1$.

Now we use this result to prove the theorem.

Since E^s_x can be regarded as graph of an element $\omega \in L(F^s_x, F^{cu}_x)$ with $\|\omega\|$ as close to zero as desired, we can assume without loss of generality that $\|\omega\| \leq 1$, and hence E^s is identified with a continuous Df-invariant section of $\mathcal{D}_1$.

Note that $Dh(x)\,\mathrm{graph}(\ell) = \mathrm{graph}(\Gamma_x(\ell))$ for $\ell \in \mathcal{D}_x$. Since $h = df^{-1}$, it follows that $E^s : U_1 \to \mathcal{D}_1$ is a continuous Γ-invariant section.

From the lemma, the graph transform $\Gamma : \mathcal{D}_1 \to \mathcal{D}_0$ defines a fiber contraction over the overflowing diffeomorphism $h : U_1 \to U_0$, and this fiber contraction is *q-sharp* in the terminology of Hirsch-Pugh-Shub (HPS).

When q is an integer, we have verified the hypotheses of the "C^r Section Theorem 3.5" from HPS (with q playing the role of r, and vector bundles replaced by disk bundles as in a Remark at p. 36 of HPS).

It follows that $E^s : U_1 \to \mathcal{D}_1$ is the unique continuous Γ-invariant section and moreover that this section is C^q.

This completes the proof in the case that q is an integer.

The general case follows from Remark 2 in p. 38 of HPS.

(Skip the proof of the lemma)

Proof of the q-sharp graph transform lemma

To prove the lemmma we start noting that by the bunching assumption, we can choose $\lambda_x \in (0,1)$ s.t.

$$\|Df \mid E^s_x\| \cdot \|Df^{-1} \mid E^{cu}_{fx}\| < \lambda_x \quad \text{and} \quad \lambda_x \, \|Df \mid T_x M\|^q < 1,$$

for all $x \in U_0$. Since f is C^1 and $\overline{U_0}$ is compact, there exists $\delta \in (0,1)$ such that $(\lambda_{hx} + 2\delta)(1-\delta)^{-2} < 1$ and

$$(\lambda_{hx} + 2\delta)(1-\delta)^{-2} \|Dh^{-1} \mid T_{hx} M\|^q < 1,$$

for all $x \in U_0$.

Since F^s is close to the Df-invariant contracting bundle E^s, we can arrange that $\|C_x\| \leq 1$ and $\|A_x^{-1}\| \leq 1$ for all $x \in U_1$.

Also, F^{cu} is close to E^{cu} which is invariant when restricted to Λ so we can arrange that $\|B_x\| < \delta$.

Moreover, A_x^{-1} is close to $Df \mid E^s_{hx}$ and D_x is close to $Df^{-1} \mid E^{cu}_x$ so we can ensure that $\|A_x^{-1}\| \|D_x\| \leq \lambda_{hx}$ for all $x \in U_1$.

Let $\ell, \ell' \in \mathbb{D}_x$. Note that $\|A_x^{-1} B_x \ell\| \leq \delta$, so $\|(I + A_x^{-1} B_x \ell)^{-1}\| \leq (1-\delta)^{-1}$. Similarly, $\|(I + A_x^{-1} B_x \ell')^{-1}\| \leq (1-\delta)^{-1}$. Hence

$$\begin{aligned}
&\|(A_x + B_x \ell)^{-1} - (A_x + B_x \ell')^{-1}\| \\
&\qquad = \|(A_x + B_x \ell)^{-1} (B_x(\ell' - \ell))(A_x + B_x \ell')^{-1}\| \\
&\qquad \leq \|A_x^{-1}\|^2 \delta (1-\delta)^{-2} \|\ell' - \ell\| \\
&\qquad \leq \|A_x^{-1}\| \delta (1-\delta)^{-2} \|\ell' - \ell\|.
\end{aligned}$$

Thus we arrive at

$$\begin{aligned}
\|\Gamma_x(\ell) - \Gamma_x(\ell')\| &\leq \|D_x(\ell - \ell')\| \|(A_x + B_x \ell)^{-1}\| \\
&\quad + \|(C_x + D_x \ell')\| \|(A_x + B_x \ell)^{-1} - (A_x + B_x \ell')^{-1}\| \\
&\leq \|A_x\|^{-1} \|D_x\| (1-\delta)^{-1} \|\ell - \ell'\| \\
&\quad + (1 + \|D_x\|) \|A_x^{-1}\| \delta (1-\delta)^{-2} \|\ell - \ell'\| \\
&\leq \lambda_{hx} (1-\delta)^{-1} \|\ell - \ell'\| + 2\delta (1-\delta)^{-2} \|\ell - \ell'\|,
\end{aligned}$$

and so

$$\mathrm{Lip}(\Gamma_x) \leq (\lambda_{hx} + 2\delta)(1-\delta)^{-2},$$

for all $x \in U_1$.

In particular, $\mathrm{Lip}(\Gamma_x) < 1$ so $\Gamma_x(\mathcal{D}_x) \subset \mathcal{D}_{hx}$, and hence Γ is well-defined.

The statement of the lemma follows from this estimate combined with

$$(\lambda_{hx} + 2\delta)(1-\delta)^{-2} \|Dh^{-1} \mid T_{hx} M\|^q < 1.$$

3.4 Smooth foliation: strong dissipativity

Strong dissipative condition

This is a verifiable condition for smoothness of stable foliations and we can get an estimate for the degree of smoothness of the stable foliation for the Lorenz attractor.

Recall that $d_s = \dim E^s_x$. Given $A = \{a_{ij}\} \in \mathbb{R}^{d\times d}$, let $\|A\|_2 = (\sum_{ij} a_{ij}^2)^{1/2}$.

Definition

Let $q > 1/d_s$. A partially hyperbolic attractor Λ is *q-strongly dissipative* if

(a) For every equilibrium $p \in \Lambda$ (if any), the eigenvalues $\lambda_1 \le \lambda_2 \le \cdots \le \lambda_d$ of $DG(p)$ satisfy $\lambda_1 - \lambda_{d_s+1} + q\lambda_d < 0$.

(b) $\sup_{x\in\Lambda}\{\operatorname{div} G(x) + (d_s q - 1)\|(DG)(x)\|_2\} < 0$.

Smooth stable foliation

Theorem

Let Λ be a sectional hyperbolic attractor. Suppose that Λ is q-strongly dissipative for some $q \in (1/d_s, [r]\,]$. Then there exists a neighborhood U_0 of Λ such that the stable manifolds $\{W^s_x,\, x \in U_0\}$ define a C^q foliation of U_0.

To prove this, for each $t \in \mathbb{R}$, we define $\eta_t : \Lambda \to \mathcal{R}$,

$$\eta_t(x) = \log\left\{\|DX^t|E^s_x\| \cdot \|DX^{-t}|E^{cu}_{X^t x}\| \cdot \|DX^t|E^{cu}_x\|^q\right\}$$

Note that $\{\eta_t,\ t \in \mathbb{R}\}$ is a continuous family of continuous functions each of which is subadditive, that is, $\eta_{s+t}(x) \le \eta_s(x) + \eta_t(X^s x)$.

Proof of smoothness condition

Let $\mathcal{M}$ denote the set of flow-invariant ergodic probability measures on Λ.

We claim that for each $m \in \mathcal{M}$, the limit $\lim_{t\to\infty} \frac{1}{t}\eta(x)$ exists and is negative for m-almost every $x \in \Lambda$.

Proposition (Arbieto-Salgado, 2010)

Let $\{t \mapsto f_t : \Lambda \to \mathbb{R}\}_{t\in\mathbb{R}}$ be a continuous family of continuous functions which is subadditive and suppose that $\int \widetilde{f}(x)d\mu < 0$ for every $\mu \in \mathcal{M}_X$, with $\widetilde{f}(x) := \lim\limits_{t\to+\infty} \frac{1}{t} f_t(x)$. Then there exist a $T > 0$ and a constant $\lambda < 0$ such that for every $x \in \Lambda$ and every $t \ge T$:

$$f_t(x) \le \lambda t.$$

It then follows that there exists constants $C, \beta > 0$ such that $\exp \eta_t(x) \le Ce^{-\beta t}$ for all $t > 0$, $x \in \Lambda$.

In particular, for t sufficiently large, $\exp \eta_t(x) < 1$ for all $x \in \Lambda$.

Hence the q-bunching condition is satisfied for such t and the result follows from the previous theorem and remarks.

It remains to verify the claim. For each $m \in \mathcal{M}$, we label the Lyapunov exponents

$$\lambda_1(m) \le \lambda_2(m) \le \cdots \le \lambda_d(m).$$

Since Λ is partially hyperbolic, the Lyapunov exponents $\lambda_j(m)$, $j = 1, \ldots, d_s$ are associated with E^s and are negative, while the remaining exponents are associated with E^{cu}.

For m-a.e. $x \in \Lambda$ we have

$$\begin{aligned}
&\lim_{t\to\infty} \frac{1}{t} \log \|DX^t|E^s_x\| = \lambda_1(m),\\
&\lim_{t\to\infty} \frac{1}{t} \log \|DX^{-t}|E^{cu}_{X^t x}\| = -\lambda_{d_s+1}(m),\\
&\lim_{t\to\infty} \frac{1}{t} \log \|DX^t|E^{cu}_x\| = \lim_{t\to\infty} \frac{1}{t} \log \|DX^t \mid T_x M\| = \lambda_d(m).
\end{aligned}$$

Hence, m-almost everywhere,

$$\lim_{t\to\infty} \frac{1}{t} \eta_t(x) = \lambda_1(m) - \lambda_{d_s+1}(m) + q\lambda_d(m).$$

If m is a Dirac delta at an equilibrium $p \in \Lambda$, then it is immediate from item (a) of the definition of strong dissipativity that $\lim_{t\to\infty} \frac{1}{t}\eta_t(p) < 0$.

If m is not supported on an equilibrium, then there is a zero Lyapunov exponent in the flow direction. **Sectional expansion ensures that** $\lambda_{d_s+1}(m) = 0$ and $\lambda_j(m) > 0$ for $j = d_s + 2, \ldots, d$. Hence, m-almost everywhere,

$$\begin{aligned}
&\lim_{t\to\infty} \frac{1}{t}\eta_t(x) = \lambda_1(m) + q\lambda_d(m) \le \frac{1}{d_s} \sum_{j=1}^{d_s} \lambda_j(m) + q\lambda_d(m)\\
&= \frac{1}{d_s}\Big(\sum_{j=1}^{d_s} \lambda_j(m) + d_s q \lambda_d(m)\Big) \le \frac{1}{d_s}\Big(\sum_{j=1}^{d} \lambda_j(m) + (d_s q - 1)\lambda_d(m)\Big)\\
&= \frac{1}{d_s} \lim_{t\to\infty} \frac{1}{t}\Big(\log|\det DX^t(x)| + (d_s q - 1)\log\|DX^t(x)\|\Big)\\
&\le \frac{1}{d_s} \lim_{t\to\infty} \frac{1}{t} \int_0^t \Big(\operatorname{div} DG(X^s x) + (d_s q - 1)\|DG(X^s x)\|_2\Big)\, ds\\
&\le d_s^{-1} \sup_{x\in\Lambda} \big\{ \operatorname{div} DG(x) + (d_s q - 1)\|DG(x)\|_2 \big\}.
\end{aligned}$$

By item (b) of the definition of strong dissipativity, we again have that $\lim_{t\to\infty} \frac{1}{t}\eta_t(x) < 0$ for m-almost every $x \in \Lambda$.

This completes the proof of the claim and the theorem follows.

4 Smoothness of stable foliation and holonomies

4.1 Smoothness estimates

$C^{1+\varepsilon}$ **stable foliation for dissipative singular-hyperbolic attracting sets**

Using the strong dissipativity and bunching results we estimate the degree of smoothness of the stable foliation for the Lorenz attractor in the classical parameters

$C^{1+\varepsilon}$ stable foliation for dissipative singular-hyperbolic attracting sets

Note that if $\sup_\Lambda \operatorname{div} G < 0$, then condition (b) holds for $q = d_s^{-1} + \varepsilon$ for ε sufficiently small.

When $\dim M = 3$, we have $d_s = 1$ and hence we deduce that in the dissipative case singular-hyperbolic attracting sets have a uniformly contracting (stable) foliation on a full neighborhood of the set and which is $C^{1+\varepsilon}$-smooth, that is, it admits $C^{1+\alpha}$ foliated charts and the holonomies along the stable leaves are also $C^{1+\varepsilon}$ for some $\varepsilon > 0$.

In the case of the Lorenz attractor in the classical parameters, we can estimate de value of $1+\varepsilon$ as follows.

$C^{1+\varepsilon}$-smooth stable foliation for the Lorenz attractor

The classical Lorenz equations

$$\begin{aligned}
\frac{dx}{dt} &= \sigma(y-x) & \sigma &= 10 \\
\frac{dy}{dt} &= rx - y - xz & r &= 28 \\
\frac{dz}{dt} &= xy - bz & b &= 8/3
\end{aligned}$$

define a smooth vector field G such that

$$\operatorname{div} G \equiv -\tfrac{41}{3}, \quad \lambda_1 \approx -22.83, \quad \lambda_2 = -\tfrac{8}{3}, \quad \lambda_3 \approx 11.83,$$

are the divergence and the eigenvalues of DG at the unique singularity at the origin, respectively.

Estimate for the degree of smoothness

Thus, since after the work of W. Tucker (2000) the classical Lorenz attractor is a geometric Lorenz attractor, we have that it is $(1+\varepsilon)$-strongly dissipative for $\varepsilon > 0$ sufficiently small.

Hence, the stable foliation is $C^{1+\varepsilon}$ for the classical Lorenz attractor, for some $\varepsilon > 0$. In fact, we can prove

Corollary

The stable foliation for the classical Lorenz attractor is at least $C^{1.264}$.

Proof of the estimate

Note that By definition, q-strong dissipativity holds for any $q < \min\{q_1, q_2\}$ where

$$q_1 = \frac{\lambda_2 - \lambda_1}{\lambda_3} \approx 1.704,$$
$$q_2 = 1 - \frac{\operatorname{div} G}{\sup_\Lambda \|DG\|_2} = 1 + \frac{41}{3} \frac{1}{\sup_\Lambda \|DG\|_2}.$$

Now

$$\|DG(x)\|_2^2 = 201 + \frac{64}{9} + 2x_1^2 + x_2^2 + (x_3 - 28)^2 \approx 208.11 + V,$$

where

$$V = 2x_1^2 + x_2^2 + (x_3 - 28)^2.$$

Estimate on the size of attracting set

To estimate $\sup_\Lambda \|DG\|_2$ there are various explicit estimates on the Lorenz basin of attraction.

One of the best and easier to state estimates can be found in Giacomini-Neukirch (1997) ["Integrals of motion and the shape of the attractor for the Lorenz model." *Phys. Lett. A*], which shows that a trapping region is given by ellipsoids of the form

$$\frac{c-28}{10} x_1^2 + x_2^2 + (x_3 - 28)^2 = R,$$

provided $R \geq \frac{c^2 b^2}{4(b-1)}$ where $b = 8/3$.

Taking $c = 48$ we obtain $\frac{c^2 b^2}{4(b-1)} = 2457.6$ and then we can explicitly calculate $V \leq 2457.6$, and so $q_2 > 1.264$ as stated.

4.2 Hölder-C^1 stable holonomies

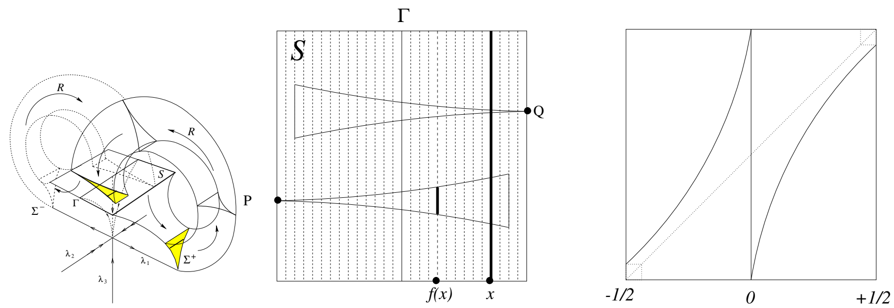
Hölder-C^1 condition on the stable holonomies

In general, even without bunching or strong dissipative condition, for singular-hyperbolic (three-dimensional) flows, using the low codimension of the stable leaves inside cross-sections, **the holonomy along stable manifolds is differentiable and its derivaties are Hölder continuous**.

Moreover, using this Hölder-C^1 property of stable holonomies, we can also show that **the Poincaré return time function to a cross-section is Hölder-continuous**.

This is used in a crucial way to study the ergodic theory of singular-hyperbolic attractors: to prove the existence of physical/SRB measure for the flow on these attractors and study its statistical properties. However the proof of these properties was only sketched in the literature.

$C^{1+\alpha}$ stable holonomies and $C^{1+\alpha}$ quotient map



Partial hyperbolic attracting set with codimension 2 stable direction

Let G be a flow on a manifold M which is partially hyperbolic on a compact invariant attracting set Λ and the stable direction has codimension 2, that is, there exists a DX_t-invariant and continuous splitting $T_\Lambda M = E^s_\Lambda \oplus E^c_\Lambda$ such that there are $C, \lambda > 0$ satisfying for every $x \in \Lambda$ and $t > 0$

- E^s is uniformly contracted: $\|DX_t \mid E^s_x\| \le Ce^{-\lambda t}$;
- E^c_Λ dominates E^s_Λ: $\|DX_t \mid E_x\| \cdot \|DX_{-t} \mid E^c_{X_t(x)}\| < Ke^{-\lambda t}$.
- if $d_s = \dim E^s_\Lambda, d^c = \dim E^c_\Lambda$ and $d = \dim M = d^s + d^c$, then $d^c = 2$ and $d^s = d - 2$.

We assume from now on that $\Lambda = \bigcap_{t>0} \overline{X^t(U_0)}$ for an open neighborhood U_0 of Λ in M.

Extensions of the stable bundle and central-unstable cone field.

We also assume that the splitting has been extended to a continuous decomposition of $T_{U_0}M = E^s \oplus E^c$ where E^s is DX^t-invariant for $t > 0$ and there exists a continuous family $(\mathcal{C}^{cu}_x)_{x \in U_0}$ of central unstable cones so that $E^c_x \subset \mathcal{C}^u_x$ and $E^s_x \cap \mathcal{C}^{cu}_x = \{\vec{0}\}$ for all $x \in U_0$.

Now let $\Sigma \subset U_0$ be a ***cross-section*** to the flow, that is, a C^2 embedded compact disk transverse to G at every point $x \in \Sigma$. Set $\tau_0 = \inf\{|t| : X^t x \in \Sigma, t \neq 0\}$, which is strictly positive by compactness of Σ.

For $x \in \Sigma$ we **define** $W^s_x(\Sigma)$ to be the connected component of $\Sigma \cap \big(\bigcup_{|t| \le \tau_0/2} X^t(W^s_x)\big)$ which contains x. **This is the stable foliation on the cross-section.**

Codimension one stable foliation on Σ

Note that because E^s is always Hölder-continuous on U_0 then W^s_X is a $C^{1+\varepsilon}$ immersed smooth submanifold of U_0, for some $\varepsilon > 0$.

In addition, since Σ and $\big(\bigcup_{|t| \le \tau_0/2} X^t(W^s_x)\big)$ are codimension one submanifolds of class $C^{1+\varepsilon}$ of U_0 which are, moreover, transverse by construction, then its intersection $W^s_x(\Sigma)$ **is a codimension one submanifold of Σ. These leaves form a codimension one foliation $\mathcal{F}^s_\Sigma$ of Σ.**

Let γ_0, γ_1 **be a pair of smooth curves contained in** Σ given by $\gamma_i : [0,1] \to \Sigma, i = 0,1$ **whose tangent space is everywhere contained in the center-unstable cone**: for some small $a > 0$

$$\gamma_i'(t) \in \mathcal{C}^{cu}_{\gamma_i(t)}(a) \cap T_{\gamma_i(t)}\Sigma, \quad \text{for all } t \in [0,1], i = 0,1.$$

Hölder-C^1 stable holonomy on cross-sections

We further assume that γ_i crosses Σ, that is, $\gamma_i([0,1]) \pitchfork W^s_x(\Sigma) = \gamma_i([0,1]) \cap W^s_x(\Sigma)$ is a single point for all $x \in \Sigma, i = 0,1$.

Hence there exists a map $h : \gamma_0 \to \gamma_1$ associating to each $\gamma_0(t)$ the unique (transversal) intersection point of $W^s_{\gamma_0(t)}(\Sigma)$ with γ_1; this is the holonomy map of $\mathcal{F}^s(\Sigma)$ from γ_0 to γ_1.

Theorem

The holonomy h is differentiable and its derivative is Hölder.

To prove this we need to consider the holonomies of the stable foliation $\mathcal{F}^s$ of the flow.

Holonomies on the cross-section and on U_0

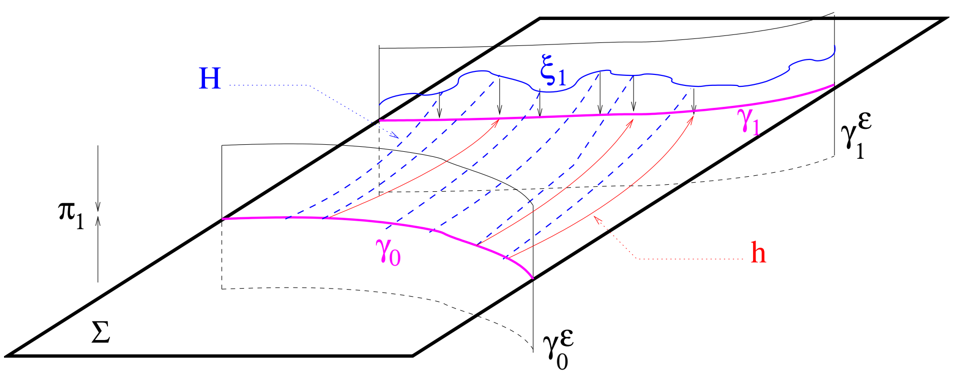


Figure 2: The cross-section Σ to the flow together with the curves γ_i and surfaces $\gamma_i^\varepsilon, i = 0,1$, the holonomy H (along the stable leaves of the flow) restricted to γ_0 and the holonomy h (along the stable leaves on the cross-section) after composing with the projection π_1.

Consequence of the Theorem

A consequence of the theorem on Hölder-C^1 smoothness of the stable holonomy on cross-sections is that **if we consider the quotient map of a Poincaré map to the cross-section Σ over the stable foliation $\mathcal{F}^s(\Sigma)$, then this quotient map becomes a $C^{1+\varepsilon}$ one-dimensional map for some $\varepsilon > 0$.**

This is the crucial feature that enables us to use the ergodic theory of one-dimensional dynamics to study the ergodic theory of these attracting sets without assuming bunching or dissipative conditions.

(Skip the proof of the theorem)

The stable holonomy for the flow on U_0

We consider the surfaces $\gamma_i^\varepsilon = \bigcup_{t\in[-\varepsilon,\varepsilon]} X^t(\gamma_i), i = 0, 1$ (at least of class C^2 since both γ_0 and X_t belong to this class) for some fixed $0 < \varepsilon < \tau_0/2$.

These are transverse to the stable foliation $\mathcal{F}^s$ of the flow, by construction.

We can then consider the holonomy $H : \gamma_0^\varepsilon \to \gamma_1^\varepsilon$ given for each $z \in \gamma_0^\varepsilon$ by the unique (transversal) intersection of W_z^s with γ_1^ε.

Proof of the Theorem

We write h as a composition of the restriction $\tilde{h} = H \mid_{\gamma_0}: \gamma_0 \to \xi_1 = H(\gamma_0) \subset \gamma_1^\varepsilon$ with $\pi_i : \gamma_i^\varepsilon \to \gamma_i, i = 0, 1$, which is the natural projection along flow lines. That is $h = \pi_1 \circ \tilde{h}$ where we set

$$\pi_1(z) = \gamma_1(s) \iff \exists |t| < \varepsilon : X^t(\gamma_1(s)) = z.$$

for some $s \in [0, 1]$.

Then we can write the image $\xi_1 = \tilde{h}(\gamma_0)$ as the following graph in γ_1^ε over γ_1:

$$\xi_1 = \{X^{\xi(\gamma_1(s))}(\gamma_1(s)) : s \in [0, 1]\}$$

for a map $\xi : \gamma_1 \to \mathbb{R}$.

Remember that $\tilde{h}$ is given by the restriction $H \mid \gamma_0$.

Now Hölder continuity of the holonomy maps H along strong-stable laminations is a general feature of $C^{1+\alpha}$ partially hyperbolic dynamics for any $\alpha > 0$; see **Pugh-Shub-Wilkinson "Hölder foliations". Duke Math. J. '97**.

Hence $\xi : \gamma_1 \to \mathbb{R}$ is Hölder-continuous because $[0, 1] \ni s \mapsto \xi_1(s) = X^{\xi(\gamma_1(s))}(\gamma_1(s))$ is a Hölder continuous curve in γ_1^ε and $(t, s) \mapsto X^t(\gamma_1(s))$ is a C^1 parametrization of the surface $\gamma_1^\varepsilon \supset \xi_1$.

Moreover, in this setting, **these holonomies are also absolutely continuous with respect to the induced smooth measures m_i on $\gamma_i^\varepsilon, i = 0, 1$ from the Riemannian volume on M**; see Pesin-Sinai "Gibbs measures for partially hyperbolic attractors" ETDS '82 or Pugh-Shub "Ergodic Attractors" TAMS '89. **This means that $H_*(m_0) \ll m_1$.**

Hölder Jacobians

This also means that H admits a Jacobian, that is, there exists $JH : \gamma_0^\varepsilon \to [0, +\infty)$ such that $m_1(H(A)) = \int_A JH \, dm_0$ for all Borel subsets A of γ_0^ε.

In addition, **this Jacobian is a Hölder-continuous map**; see e.g. Theorem 8.6.13, p 255 in Barreira-Pesin "Nonuniform hyperbolicity" CUP '07.

Let us denote by λ_i the measure induced on γ_i by the area measure m_i from γ_i^ε, $i = 0, 1$.

Altogether this ensures that $h : \gamma_0 \to \gamma_1$ is absolutely continuous in the sense that $h_*(\lambda_0) \ll \lambda_1$ and its Jacobian is also Hölder-continuous, which implies that the Radon-Nikodym derivative $\frac{d(h_*\lambda_0)}{d\lambda_1}$ can be seen as λ_1-a.e. equal to h', and so h becomes a Hölder-C^1 map!

Holonomy has derivative which is Hölder

Indeed, given any open interval $(a, b) \subset [0, 1]$ we define $\lambda_i(\gamma_i(a, b)) = m_i(\pi_i^{-1}\gamma_i(a, b))$, $i = 0, 1$ and so

$$\begin{aligned}
\lambda_1\big(h(\gamma_0(a,b))\big) &= \lambda_1\big(\pi_1\tilde{h}(\gamma_0(a,b))\big) = \lambda_1\big(\pi_1 H(\pi_0^{-1}\gamma_o(a,b))\big) \\
&= m_1\big(H(\pi_0^{-1}\gamma_0(a,b))\big) \\
&= \int_{\pi_0^{-1}\gamma_0(a,b)} JH\, dm_0 = \int_{\gamma_0(a,b)} JH\, d((\pi_0)_* m_0) \\
&= \int_{\gamma_0(a,b)} JH\, d\lambda_0
\end{aligned}$$

we see that the Jacobian of h can be seen as the restriction of JH to the image of γ_0.

Absolute continuity and a.e. differentiability

Finally, **absolutely continuous maps as h are differentiable λ_0-a.e., that is h' exists λ_0-a.e. and, moreover, are primitives of the derivative.** So we have

$$\lambda_1\big(h(\gamma_0(a,b))\big) = \int_{\gamma_0(a,b)} |h'|\, d\lambda_0$$

for all $0 \le a < b \le 1$.

Since we also know that $|h' \circ \gamma_0| = JH \circ \gamma_0$, λ_0-a.e. and JH is Hölder-continuous, then we can extend h' to a Hölder-continuous function $[0, 1] \to \mathbb{R}$ which is the derivative of h.

This concludes the proof of the Hölder-C^1 smoothness of holonomies in this setting.

4.3 Piecewise expansion

Piecewise expansion for the quotient map

If we also assume that E^c is seccionally expanding, then we can find a collection of cross-sections to the flow and a Poincaré return map which admits a one-dimensional quotient map over the stable foliation that is a $C^{1+\varepsilon}$ piecewise expanding map.

Cross-sections and Poincaré maps

Given two cross-sections $\Sigma, \widetilde{\Sigma}$ to the flow, let us assume that there exists $x \in \mathrm{int}(\Sigma)$ and $\tau > 0$ so that $X_\tau(x) \in \mathrm{int}(\widetilde{\Sigma})$ (we write $\mathrm{int}(\Sigma)$ for the interior of Σ as a manifold with boundary).

The Tubular Flow Theorem ensures that there exists an open neighborhood U_x of x in Σ and a uniquely defined smooth *Poincaré map*

$$f : U_x \subset \Sigma \to \widetilde{\Sigma}, \quad r(x) = X_{r(x)}(x) \tag{1}$$

for a suitable *Poincaré return time* function $r : U_x \to \mathbb{R}^+$ with $r(x) = \tau$, in such a way that $f\mid_{U_x}$ becomes a *diffeomorphism onto* an open neighborhood $V_{fx} = f(U_x)$ of fx in $\widetilde{\Sigma}$ and as smooth as the vector field G.

Holonomies on cu-curves

Note that, in general, f needs not correspond to the first time the orbits of $U_x \subset \Sigma$ encounter $\widetilde{\Sigma}$, nor it is defined everywhere in Σ.

Note that the return time function $r : \Sigma \to (0, +\infty)$ belongs to the same differentiability class as the flow, since the cross-sections $\Sigma, \widetilde{\Sigma}$ are smooth embedded disks on M.

Let us assume that $\Sigma, \widetilde{\Sigma}$ are endowed with cu-curves $\gamma_0, \widetilde{\gamma_0}$ which cross each cross-section and also U_x and V_{fx}, respectively.

We denote $p : U_x \to \gamma_0$, $p' : V_{fx} \to \widetilde{\gamma_0}$ the projections along the stable foliation $\mathcal{F}^s_\Sigma$ and $\mathcal{F}^s_{\widetilde{\Sigma}}$ on each neighborhood.

Locally quotienting over the stable foliation

The open ngbh. U_x where f is defined projects onto $V = p(U_x)$ which is an open neighborhood of $p(x)$ in γ_0. Since stable leaves are invariant, we can define

$$y \in V \mapsto \bar{f}(y) = p'\big(f(p^{-1}(y) \cap U_x)\big) \in \widetilde{\gamma_0}.$$

From previous results, this is a composition of a $C^{1+\alpha}$ map with the Poincaré map, and thus $\bar{f}$ is a $C^{1+\alpha}$ map, for some $0 < \alpha < 1$.

If we have that

- f is defined on all points of Σ, and that
- **f sends leaves of $\mathcal{F}^s_\Sigma$ into the interior of leaves of $\mathcal{F}^s_{\widetilde{\Sigma}}$;**

then, taking the cu-curves $\gamma_0, \widetilde{\gamma_0}$ crossing $\Sigma, \widetilde{\Sigma}$, respectively, the previous procedure defines a quotient map $\bar{f} : \gamma_0 \to \widetilde{\gamma_0}$ which is a $C^{1+\alpha}$ map.

Partial hyperbolicity of Poincaré maps

The splitting $E^s \oplus E^{cu}$ over U_0 induces a continuous splitting $E^s(\Sigma) \oplus E^{cu}(\Sigma)$ of the tangent bundle $T\Sigma$ (and analogously for $\widetilde{\Sigma}$)

$$E_y^s(\Sigma) = E_y^{cs} \cap T_y\Sigma \quad \text{and} \quad E_y^{cu}(\Sigma) = E_y^{cu} \cap T_y\Sigma, y \in \Sigma$$

where $E_y^{cs} = E_y^s \oplus E_y^G$ and E_y^G is the direction of the flow at y.

The DX_t-invariance of the splitting $E^s \oplus E^{cu}$ on Λ and the invariance of E^s on U_0 ensures that

- $Df \cdot E_x^s(\Sigma) = E_{fx}^s(\Sigma)$ for all $x \in \Sigma$, and
- $Df \cdot E_x^{cu}(\Sigma) = E_{fx}^{cu}(\Sigma)$ for all $x \in \Lambda \cap \Sigma$.

Partial hyperbolic Poincaré map

The next result shows that, if $T_\Lambda M = E_\Lambda^s \oplus E_\Lambda^c$ is a partial hyperbolic splitting and the Poincaré time $r(x)$ is sufficiently large, then $E^s(\Sigma) \oplus E^{cu}(\Sigma)$ defines a partially hyperbolic splitting for the transformation f on the cross-sections.

Proposition

Let $f : \Sigma \to \widetilde{\Sigma}$ be a Poincaré map with Poincaré time r. For every given $0 < \lambda < 1$ there exists $T_1 = T_1(\Sigma, \widetilde{\Sigma}, \lambda) > 0$ such that if $\inf r > T_1$, then

- $\|Df \mid E_x^s(\Sigma)\| < \lambda$, and
- $\|Df \mid E_x^s(\Sigma)\| \cdot \left\| \big(Df \mid E_x^{cu}(\Sigma)\big)^{-1} \right\| < \lambda$

for all $x \in \Sigma$.

Proof of the proposition

Note that for $v \in T_x\Sigma$ we have

$$Df(x)v = D(X_{r(x)}(x))v = DX_{r(x)} \cdot v + \big(Dr(x) \cdot v\big)G(fx) \in T_{fx}\widetilde{\Sigma}$$

which is the same as

$$Df(x)v = \pi_{\widetilde{\Sigma}}(fx) \cdot \big(DX_{r(x)} \cdot v\big)$$

where $\pi_{\widetilde{\Sigma}}(fx) : T_{fx}M \to T_{fx}\widetilde{\Sigma}$ is the projection corresponding to the splitting $T_{fx}M = T_{fx}\Sigma \oplus \big(\mathbb{R} \cdot G(fx)\big)$.

Since $\pi_{\widetilde{\Sigma}}(z)$ has uniformly bounded norm for $z \in \widetilde{\Sigma}$ by compactness and transversality, then the statement of the proposition is a straightforward consequence of partial hyperbolicity, as long as r is big enough.

Standard parametrization for cross-sections

In this way we can always achieve an arbitrarily large contraction rate along the stable direction at any given pair of cross-sections, as long as we take λ sufficiently close to zero and, consequently, a big enough threshold time T_1.

Given a cross-section Σ there is no loss of generality in assuming that it is the image of the square I^2 by a $C^{1+\alpha}$ diffeomorphism h, for some $0 < \alpha < 1$, which sends vertical lines inside leaves of $\mathcal{F}^s(\Sigma)$, where $I = [-1, 1]$. We denote by $\mathrm{int}(\Sigma)$ the image of $\mathrm{int}(I^2) = (-1, 1)^2$ under the above-mentioned diffeomorphism, which we call the *interior* of Σ.

We also say that $\partial I \times I \simeq \partial^u \Sigma$ is the unstable-boundary of Σ and that $I \times \partial I \simeq \partial^s \Sigma$ is the stable-boundary of Σ. Notice that $\partial^s \Sigma$ is formed by two curves inside the stable foliation.

We also assume that each cross-section Σ is contained in U_0, so that every $x \in \Sigma$ is such that $\omega(x) \subset \Lambda$. For convenience, from now on we assume that cross-sections are of this kind.

Generalized Lorenz singularity

A *generalized Lorenz singularity* is an equilibrium σ of G such that the spectrum of $DG(\sigma)$ has two largest real eigenvalues satisfying $\lambda_2 < 0 < \lambda_3$ and the rest of the spectrum is contained in $\{z \in \mathbb{C} : \Re(z) < \lambda_2\}$.

Hence such singularities have a strong-unstable one-dimensional manifold W^u_σ, a strong-stable $(d-2)$-dimensional manifold W^{ss}_σ and a stable $(d-1)$-dimensional manifold W^s_σ.

However, the derivative $DG(\sigma)$ of the flow at σ is *not necessarily area expanding along the directions corresponding to the eigenvalues* λ_2, λ_3, as is the case of a *Lorenz-like* singularity.

Cross-sections near a Lorenz-like equilibrium

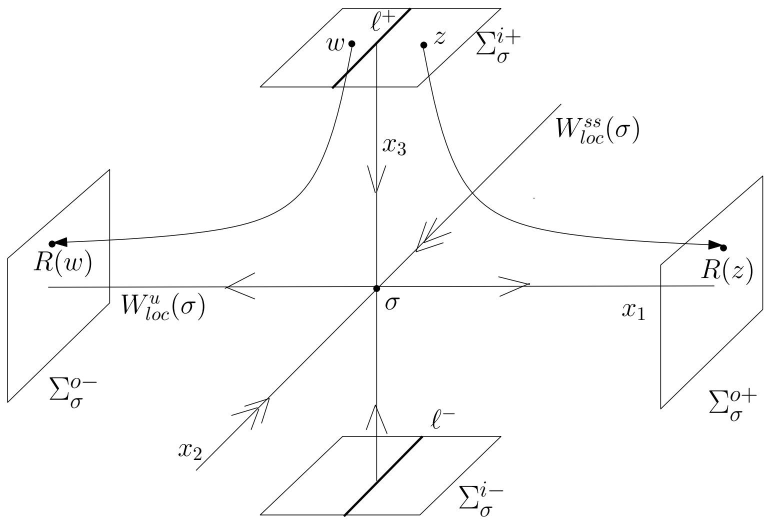
Global Poincaré map

Theorem

Let G be a C^2 vector field on a d-dimensional compact manifold having a partial hyperbolic attracting set Λ, with $T_\Lambda M = E^s_\Lambda \oplus E^c_\Lambda$ and $\dim E^s_\Lambda = d-2$, and containing generalized Lorenz singularities.

For $S(\Lambda) = \{\sigma \in \Lambda : G(\sigma) = \vec{0}\}$ we assume that $W^{ss}_\sigma \cap \Lambda = \{\sigma\}$ for all $\sigma \in S(\Lambda)$.

Then there exists $\alpha > 0$ and a finite family Ξ of cross-sections and a global (n-th return) Poincaré map $R : \Xi_0 \to \Xi$, $R(x) = X_{\tau(x)}(x)$ such that



Global Poincaré map (continued)

Theorem (continued)

1. the domain $\Xi_0 = \Xi \setminus \Gamma$ contains the cross-sections with a family Γ of finitely many smooth arcs removed and $\tau : \Xi_0 \to [\tau_0, +\infty)$ is a smooth function bounded away from zero by some uniform constant $\tau_0 > 0$.

2. We can choose coordinates on Ξ so that the map R can be written as $F : \tilde{Q} \to Q$, $F(x,y) = (f(x), g(x,y))$, where $Q = I \times I$ and $\tilde{Q} = Q \setminus \Gamma_0$, with $\Gamma_0 = \mathcal{C} \times I$ and $\mathcal{C} = \{c_1, \ldots, c_n\} \subset I$ a finite set of points.

3. The map $f : I \setminus \mathcal{C} \to I$ is piecewise $C^{1+\alpha}$ with $n+1$ strictly monotonous branches defined on the connected components of $I \setminus \mathcal{C}$.

Global Poincaré map (terminates!)

Theorem (continued again)

(4) The map $g : \tilde{Q} \to I$ preserves and uniformly contracts the vertical foliation $\mathcal{F} = \{\{x\} \times I\}_{x \in I}$ of Q: $\exists 0 < \lambda < 1$ s.t. $\operatorname{dist}(g(x,y_1), g(x,y_2)) \le \lambda \cdot |y_1 - y_2|$, $\forall y_1, y_2 \in I$.

If we assume, in addition, that E_Λ^{cu} is sectionally expanding, then we can replace item (3) above by

(5) The map $f : I \setminus \mathcal{C} \to I$ is piecewise expanding $C^{1+\alpha}$ with $n+1$ strictly monotonous branches defined on the connected components of $I \setminus \mathcal{C}$ and satisfies $|Df| > 2$ wherever defined.

Flow-boxes near equilibria

Since the equilibria σ in our setting are all Lorenz-like, using the linearization given by the Hartman-Grobman Theorem or, in the absence of resonances, the smooth linearization results provided by e.g. Sternberg, orbits of the flow in a small neighborhood U of the equilibrium are solutions of a linear vector field modulo a continuous/smooth change of coordinates.

Then for $\delta > 0$ we choose cross-sections

- $\Sigma^{o\pm}$ at points $y^\pm$ in different components of $W^u_{loc}(\sigma) \setminus \{\sigma\}$
- $\Sigma^{i\pm}$ at points $x^\pm$ in different components of $W^s_{loc}(\sigma) \setminus W^{ss}_{loc}(\sigma)$

and Poincaré first hitting time maps $R^\pm : \Sigma^{i\pm} \setminus \ell^\pm \to \Sigma^{o-} \cup \Sigma^{o+}$, where $\ell^\pm = \Sigma^{i\pm} \cap W^s_{loc}(\sigma)$, satisfying

Cross-sections near singularities

1. every orbit in the attractor passing through a small neighborhood of the equilibrium σ intersects some of the incoming cross-sections $\Sigma^{i\pm}$;
2. $R^\pm$ maps each connected component of $\Sigma^{i\pm} \setminus \ell^\pm$ diffeomorphically inside a different outgoing cross-section $\Sigma^{o\pm}$, preserving the corresponding stable foliations.

These cross-sections may be chosen to be planar relative to some linearizing system of coordinates near σ, e.g., for a $\varepsilon > 0$

$$\Sigma^{i,\pm} = \{(x_1, x_2, \pm 1) : |x_1| \leq \varepsilon, |x_2| \leq \varepsilon\} \quad \text{and}$$
$$\Sigma^{o,\pm} = \{(\pm 1, x_2, x_3) : |x_2| \leq \varepsilon, |x_3| \leq \varepsilon\},$$

where the x_1-axis is the unstable manifold near $\sigma = \vec{0}$, the x_2-axis is the strong-stable manifold and the x_3-axis is the weak-stable manifold of the equilibrium.

Cross-sections near a Lorenz-like equilibrium

Covering of Λ by flow boxes

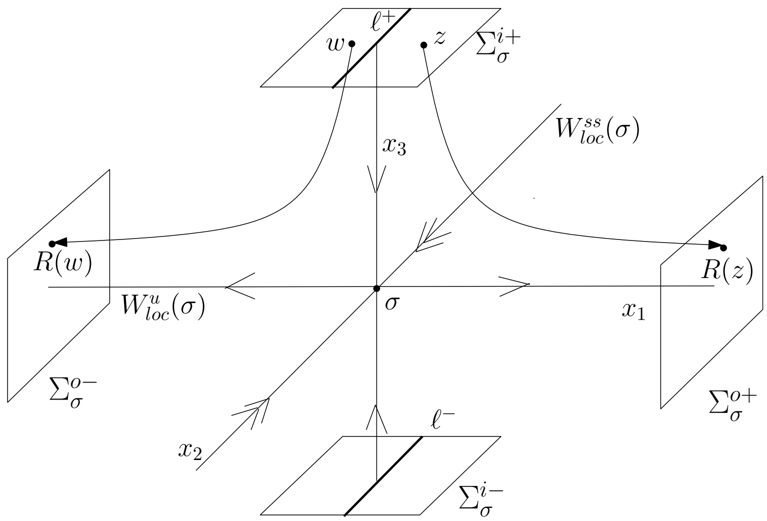
Around each singularity $\sigma \in S(\Lambda)$ there exists a flow-box covering a neighborhood U_σ of σ and at each regular point $x \in \Lambda$ there exists a cross-section Σ_x to the vector field.

Define for any cross-section Σ the δ-subsection

$$\Sigma^\delta = \{x \in \Sigma : d(x, \partial^s \Sigma) > \delta\}.$$

Take flow boxes near singularties with ingoing and outgoing subcross-sections $\Sigma^{i\pm,\delta}_\sigma, \Sigma^{o\pm,\delta}_\sigma$ covering a corresponding neighborhood U^δ_σ of $\sigma \in S(\Lambda)$ and, for each Σ_x in $\Lambda \setminus \cup_{\sigma \in S(\Lambda)} U^\delta_\sigma$ take a cross-section Σ_x to the vector field and its subsection Σ^δ_x.

Using a tubular neighborhood construction, we linearise the flow in an open set $U^\delta_\Sigma = X_{(-\varepsilon_0,\varepsilon_0)}(\mathrm{int}(\Sigma^\delta_x))$ for a small $\varepsilon_0 > 0$, containing the interior of the cross-section Σ^δ_x.



This provides an open cover of the compact set Λ by flow-boxes near the singularities and tubular neighborhoods around regular points.

We let $\Xi^\delta = \{U^\delta_{\Sigma_i}, U^\delta_{\sigma_k} : i = 1, \dots, l;\ k = 1, \dots, s\}$ be a finite cover of Λ, where $s \geq 1$ is the number of singularities in Λ, and we set $T_2 > 0$ to be an upper bound for the time it takes any point $z \in U_{\Sigma_i}$ to leave this tubular neighborhood under the flow, for any $i = 1, \dots, l$.

The global Poincaré return map

Let $T_3 = \max\{T_2, T_1(\Sigma, \widetilde{\Sigma}, \lambda), \Sigma, \widetilde{\Sigma} \in \Xi^\delta\}$ and consider the value $T > T_3$ so that

$$\operatorname{diam}\big(X_T(W^s_x(\Sigma))\big) \leq c\lambda^T \operatorname{diam}(W^s_z(\Sigma) < \frac{\delta}{100}, \quad \text{for all} \quad \Sigma \in \Xi$$

(note that here we consider $\Sigma \in \Xi$ instead of $\Sigma \in \Xi^\delta$). Then define

$$R(z) = X_{\tau(X_T(z))}\big(X_T(z)\big)$$

where $\tau(w) = \inf\{t > 0 : X_t(w) \in \Xi^\delta\}$.

Note that τ is not defined at points $w \in U_0$ which do not return to Ξ^δ, which is only possible if $X_T(w) \in W^s_{loc}(\sigma)$ for some $\sigma \in S(\Lambda)$, since the flow-boxes through the sections of Ξ^δ provide an open cover for the attracting set Λ.

The adapted Poincaré map

Let $\Xi^\delta_0 \subset \Xi^\delta$ be the set of points such that R is well-defined. By the choice of T we have that for every $x \in \Xi^\delta_0$ there exist $\Sigma, \widetilde{\Sigma} \in \Xi$ such that

$$R\big(W^s_x(\Sigma)\big) \subset \widetilde{\Sigma}^{\delta/2}.$$

This means that all points in $W^s_x(\Sigma)$ do return to $\widetilde{\Sigma}^{\delta/2}$, then we have proved

Proposition
There exists a cover of Λ by flow-boxes through cross-sections near regular points Ξ and a Poincaré return map $R : \Xi_0 \subset \Xi \to \Xi$ such that for all $x \in \Xi_0$ there are $\Sigma, \widetilde{\Sigma} \in \Xi$ such that $R\big(W^s_x(\Sigma)\big) \subset \widetilde{\Sigma}^{\delta/2}$ and so $R\big(W^s_x(\Sigma)\big) \subset \mathrm{int}\,\big(W^s_{Rx}(\widetilde{\Sigma})\big)$.

Finitely many strips in the domain of R
Now we focus of Ξ_0. Let $\partial^s\Xi$ denote the union of all the leaves forming the stable boundary of every cross-section in Ξ.

Lemma
The set of discontinuous points of R together with points where R is not defined in $\Xi \setminus \partial^s\Xi$ is contained in the set of points $x \in \Xi \setminus \partial^s\Xi$ so that

1. either $R(x)$ is defined and belongs to $\partial^s\Xi$;
2. or there is some time $0 < t \leq T$ such that $X_t(x) \in W^s_{loc}(\sigma)$ for some $\sigma \in S(\Lambda)$.

Moreover this set is contained in a finite number of stable leaves of the cross-sections $\Sigma \in \Xi$.

The global one-dimensional quotient map f
Let Γ be the finite set of stable leaves of Ξ provided by the previous lemma together with $\partial^s\Xi$. Then the complement $\Xi \setminus \Gamma \subset \Xi_0$ of this set is formed by finitely many open strips where R is smooth.

We choose a C^2 cu-curve γ_Σ transverse to $\mathcal{F}^s_\Sigma$ in each $\Sigma \in \Xi$. Then the projection p_Σ along leaves of $\mathcal{F}^s_\Sigma$ onto γ_Σ is a $C^{1+\alpha}$ map, for some $\alpha > 0$, since this is also the holonomy between cu-curves crossing $\mathcal{F}^s_\Sigma$. We set

$$J = \bigcup_{\Sigma, \widetilde{\Sigma} \in \Xi} \mathrm{int}\,\big(\{x \in \Sigma : Rx \in \widetilde{\Sigma}\}\big) \cap \gamma_\Sigma$$

which is diffeomorphic to a *finite union of non-degenerate open intervals* $I_1, \dots, I_{n+1}$ by a $C^{1+\alpha}$ diffeomorphism, and $p_\Sigma \mid p_\Sigma^{-1}(J)$ becomes a $C^{1+\alpha}$ submersion.

After rescalling we make the identification $I = \big(\cup_{i=1}^{n+1} I_i\big) \cup \mathcal{C}$, where $\mathcal{C}$ is a finite set of points in I which are boundaries of the open intervals $I_1, \dots, I_{n+1}$ in I.

Note that since Ξ is finite we can choose γ_Σ so that p_Σ has bounded derivative: there exists $\beta_0 > 1$ such that

$$\frac{1}{\beta_0} \leq \big|Dp_\Sigma \mid \gamma\big| \leq \beta_0 \textit{ for every cu-curve } \gamma \text{ inside any } \Sigma \in \Xi.$$

Since the Poincaré map $R : \Xi_0 \to \Xi$ takes stable leaves of $\mathcal{F}^s_\Sigma$ inside stable leaves of the same foliation, is hyperbolic and, in addition a cu-curve $\gamma \subset \Sigma$ is taken by R into a cu-curve $R(\gamma)$ in the image cross-section, the map

$$f : I \setminus \mathcal{C} \to I \quad \text{given by} \quad I \setminus \mathcal{C} \ni z \mapsto p_{\widetilde{\Sigma}}\Big(R\big(W^s_z(\Sigma) \cap \widetilde{\Sigma}\big)\Big)$$

for $\Sigma, \widetilde{\Sigma} \in \Xi$ is $C^{1+\alpha}$ for points in the interior of I_i, $i = 1, \dots, n+1$.

Moreover, it also satisfies

$$\big|Df\big| = \big|D\big(p_{\widetilde{\Sigma}} \circ R \circ \gamma_\Sigma\big)\big| \geq \frac{1}{\beta_0} \cdot \big\|D(R \circ \gamma_\Sigma)\big\| > 0$$

since $R(\gamma)$ is a cu-curve if γ is a cu-curve.

This completes the proof of items (1-4) of the Theorem.

The singular-hyperbolic case

We assume now the extra condition that E^c is seccionally expanded. In this setting, the singularities $S(\Lambda)$ become Lorenz-like singularities.

Given a cross-section Σ, a positive number ρ, and a point $x \in \Sigma$, we define the unstable cone of width ρ at x by

$$C^u_\rho(x) = \{v = v^s + v^u : v^s \in E^s_\Sigma(x),\ v^u \in E^{cu}_\Sigma(x) \text{ and } \|v^s\| \leq \rho \|v^u\|\}.$$

Let $\rho > 0$ be any small constant.

Hyperbolicity of Poincaré maps

Proposition

Let $R : \Sigma \to \widetilde{\Sigma}$ be a Poincaré map as before with Poincaré time $t(\cdot)$. Then $DR_x\big(E^s_x(\Sigma)\big) = E^s_{Rx}(\widetilde{\Sigma})$ at every $x \in \Sigma$ and $DR_x\big(E^{cu}_x(\Sigma)\big) = E^{cu}_{Rx}(\widetilde{\Sigma}))$ at every $x \in \Lambda \cap \Sigma$. In addition, for every given $0 < \lambda < 1$ there exists $T_3 = T_3(\Sigma, \widetilde{\Sigma}, \lambda) > 0$ such that, if $t(\cdot) > T_3$ at every point, then

$$\|DR \mid E^s_x(\Sigma)\| < \lambda \quad \text{and} \quad \|DR \mid E^{cu}_x(\Sigma)\| > 1/\lambda, \forall x \in \Sigma \cap \Lambda.$$

Moreover, any $x \in \Sigma$, we have $DR(x)(C^u_\rho(x)) \subset C^u_{\rho/2}(Rx)$ and

$$\|DR_x(v)\| \geq \frac{5}{6}\lambda^{-1} \cdot \|v\| \quad \text{for all} \quad v \in C^u_\rho(x).$$

Sketch of the proof of the proposition

The proof of this result is based on the observation that the volume expansion along the bidimensional bundle E^c_Λ translated into expansion in the $E^{cu}(\Sigma)$ direction since the vector field in invariant and non-expanding transversely to Σ.

Then, for small $\rho > 0$, the vectors in $C^u_\rho(x)$ can be written as the direct sum of a vector in E^{cu}_x, which is expanded at a rate λ^{-1}, with a vector in E^{cs}_x, which is contracted at a rate λ.

Hence, for small ρ, the center-unstable component dominates the stable component and the length of the vector is increased at a rate close to λ^{-1}.

Completing the proof of the theorem

In this way we can always achieve an arbitrarily large expansion rate along the directions of the unstable cone as long as we take λ sufficiently close to zero and, consequently, a big enough threshold time T_3.

Using this in the construction of Ξ choosing T in such a way that besides the conditions in the previous subsection, it also satisfies $T > T_3$, we obtain

$$|Df| \geq \sin \sphericalangle\big(\mathcal{F}^s_{\widetilde{\Sigma}}(R \circ \gamma_\Sigma), \gamma_{\widetilde{\Sigma}}\big) \cdot \|DR \circ \gamma_\Sigma \cdot \gamma'_\Sigma\| > \omega,$$

as long as we take the threshold time T large enough, since the angle between the cu-curves $\gamma_0, \widetilde{\gamma}_0$ and the stable foliation on the cross-sections are bounded away from zero.

This completes the proof of the Theorem.

Finally, we have reached...

THE END.

THANKS!

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4.4 Robust transitivity

Unstable cone-fields on cross-section and singular hyperbolicity

We present a proof of a claim made by Tucker in Section 2.4 of

- **W. Tucker. A rigorous ODE solver and Smale's 14th problem. *Found. Comput. Math.* 2 (2002) 53–117.**

which to the best of the authors knowledge is missing in the literature.

What Tucker proved via a computer algorithm

In the above cited paper Tucker proved, through the successful run of a computer algorithm, that there exists:

- a compact set N contained in the cross-section $\Sigma = \{z = 27\}$ of the flow G of the Lorenz equations for which:
  - the first Poincaré return map $R : N \setminus \Gamma \to N$ is well-defined away from the curve $\Gamma \subset N$, given by the intersection of the local stable manifold of the singularity with N;
  - moreover, it is proved also that $R(N \setminus \Gamma) \subset N$, so that in N there exists an attracting set $\Lambda_N = \bigcap_{n \geq 0} R^n(N)$.

The unstable cone field in the return region

In addition, there exists a cone field $\{\mathcal{C}^u_x\}_{x \in N} \subset T_N\Sigma$ s.t.

$$DR_x \mathcal{C}^u_x \subset \mathcal{C}^u_{Rx}, \quad x \in N$$

(forward invariance) and also satisfies

Proposition (Proposition 5.1 from Tucker)

There exists $F \subset N$ s.t. $F \supset \Gamma$ and contains a *fundamental domain* of R (i.e. every R-orbit has some element in F)

1. each $x_0 \in F$ whose positive orbit eventually leaves F satisfies for every return $x_n \in F$
$$\min\{\|DR^n_{x_0} \cdot v\|/\|v\| : v \in \mathcal{C}^u_{x_0}\} \geq 2;$$
2. each $x_0 \in F$ whose positive orbit is contained in F satisfies $\min\{\|DR^n_{x_0} \cdot v\|/\|v\| : v \in \mathcal{C}^u_{x_0}\} \geq 2^{n/2}$ for all $n \geq 1$.

Consequences

It follows from the algorithms developed and studied by Tucker that these are robust properties of the flow (i.e. they hold true also for all vector fields sufficiently C^1 close to G) and are enough to prove transitivity for the return map.

Lemma 2 (Transitivity lemma)**.** *For each $x \in N$ and $y \in \Lambda_N$ and open neighborhoods V of x and W of y in N, there is $m \geq 1$ s.t. $R^m V \cap W \neq \emptyset$.*

Recall that the maximal invariant subset $\Lambda = \bigcap_{t>0} \overline{X_t(U)}$ for some positively invariant neighborhood U satisfies $\Lambda \cap N = \Lambda_N$ is the maximal invariant subset at the cross-section.

Hence the above lemma implies the robust transitivity of Λ. We present a proof in what follows.

Robust transitivity and singular-hyperbolicity

Robust transitivity implies that Λ is a singular-hyperbolic attractor following

- C. A. Morales, M. J. Pacifico and E. R. Pujals. Robust transitive singular sets for 3-flows are partially hyperbolic attractors or repellers. *Ann. of Math. (2)* **160** (2004) 375–432.

From what has already been proved we get

Claim (Section 2.4 of Tucker's paper)
R admits an invariant contracting $C^{1+\alpha}$ foliation.

Existence of physical/SRB measure

Hölder-C^1 smoothness is crucial to obtain the existence of a physical/SRB measure for Λ: this ensures that the one-dimensional quotient map is a piecewise expansive $C^{1+\varepsilon}$ map for some $\varepsilon > 0$.

Then we can apply results from the ergodic theory of piecewise expanding maps of the interval, ensuring the existence of a unique absolutely continuous invariant measure ν for this map.

From this, through standard constructions of ergodic theory, a physical measure μ for the flow can be induced from the a.c.i.m. ν for the one-dimensional quotient map.

(Skip the proof of the transitivity lemma)

Proof of transitivity for the Poincaré return map

Let $N \setminus \Gamma = N^+ \cup N^-$ be the components of N away from Γ; see next figure.

There exist $\omega^\pm$ the limit points of images $R(x_n)$ when $x_n \to \Gamma$ with $x_n \in \Gamma^\pm$, due to the dynamics of the flow near the singularity at the origin.

Then we can define for $\varepsilon > 0$ and $k \in \mathbb{Z}^+$ the neighborhood of Γ in N

$$\Gamma_\varepsilon^k = \{x \in N^+ : R^k(x) \in B_\varepsilon(R^{k-1}(\omega^+))\} \\ \cup \{x \in N^- : R^k(x) \in B_\varepsilon(R^{k-1}(\omega^-))\}.$$

Before the proof: two remarks

- The previous Proposition from Tucker ensures the existence of $K > 0$ and $\sigma > 1$ such that
$$\|DR_x^n \cdot v\| \geq K\sigma^n \|v\|$$
for all $n \geq 1, v \in \mathcal{C}_x^u$ and $x \in N$ such that $R^k x \notin \Gamma$ for $k = 0, \dots, n$.

- The expansion rate provided by the same Proposition ensures that every curve $\xi : [0,1] \to N$ such that $\xi'(s) \in \mathcal{C}_{\xi(s)}^u$ **(a $\mathcal{C}^u$-curve in what follows)** admits $N = N(\xi) \in \mathbb{Z}^+$ so that $R^n\xi$ crosses N and also Γ_ε^1 for all $n \geq N$.

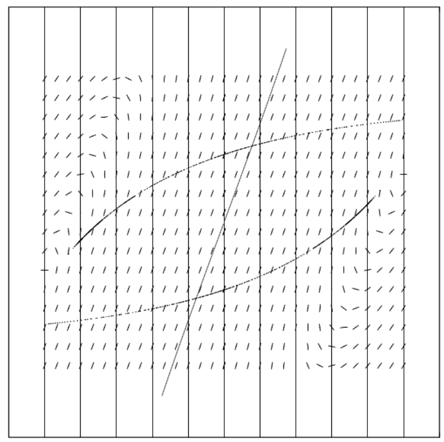

Figure 3: An approximation of Λ_N (the two curved "lines") with the most contracting directions for one iterate of R. The (almost) straight line cutting across the two branches of Λ_N is Γ, the intersection of the stable manifold of the origin and the return plane. The bounding box is $[-6,6]^2 \times \{27\}$.

Proof of the transitivity lemma

Let $y \in \Lambda_N$ and $x \in N$ be given and fix neighborhoods V of x and W of y in N.

Fix also a $\mathcal{C}^u$-curve $\xi : [0,1] \to V$ containing x.

From the previous remarks, consider $n > 0$ such that a neighborhood $V_0 \subset V$ of x satisfies that $R^n(V_0 \cap \xi)$ contains a curve ζ which crosses N and in particular crosses Γ^1_ε.

Let $\varepsilon > 0$ be small enough so that $B_{3\varepsilon}(y) \subset W$.

We split the argument in two cases, as follows.

Case A *For $z \in B_\varepsilon(y) \cap \Lambda_N$ and $z_k \in \Lambda_N$ so that $R^k z_k = z$, then $z_k \in N \setminus \Gamma^k_\varepsilon, \forall k \geq 1$.*

Case A

The assumption ensures that $W_k = R^{-k} W \subset N \setminus \Gamma^k_\varepsilon$ is diffeomorphic to W for $k = 1, \dots, \ell$ for some maximal $\ell \geq 1$.

Note that ℓ can be made arbitrarily big by reducing the size of the neighborhood W.

Let $\eta : [0,1] \to W$ be a $\mathcal{C}^s$-curve, that is, a regular curve such that $\eta'(s) \in \mathcal{C}^s_{\eta(s)} = T_{\eta(s)}\Sigma \setminus \overline{\mathcal{C}^u_{\eta(s)}}$ for all $0 \leq s \leq 1$.

The forward invariance of the cone field $\mathcal{C}^u$ implies the backward invariance of the interior of its complement $\mathcal{C}^s$, which is also a cone field.

Hence $\eta_k = R^{-k}\eta$ is also a $\mathcal{C}^s$-curve.

R **forward contracts area uniformly**

Since $\operatorname{div} G \le -c < 0$ for a constant $c > 0$ there is $C > 0$ and $0 < \lambda < 1$ s.t. $|\det DR^j| \le C\lambda^j$ for $j \ge 0$.

Indeed, since $N \subset \Sigma$ is a cross-section to the flow G, if $x \in N$ and $R(x) \in N$ is given by $X^{\tau(x)}(x)$, where $\tau(x)$ is the Poincaré return time to N, then

$$e^{-c\tau(x)} = |\det DX^\tau(x)_x| = |\det DR_x| \frac{\sin \sphericalangle(G(Rx), T_{Rx}\Sigma)}{\sin \sphericalangle(G(x), T_x\Sigma)} \\ \ge C|\det DR_x|.$$

Since $\tau(x) \ge \tau_0 > 0$ for all $x \in N$ by compactness, the uniform contraction of area of R is clear.

η_k **is forward contracted at a uniform rate**

By the backward invariance of the stable cones, there exists $\theta > 0$ for which $\sphericalangle(\eta_k'(s), v) \ge \theta$ for all $s \in [0,1], v \in \mathcal{C}^u_{\eta_k(s)}$ and $1 \le k \le \ell$. We deduce

$$|\det DR^k_{\eta_k(s)}| = \frac{\|DR^k_{\eta_k(s)}\eta_k'(s)\| \cdot \|DR^k_{\eta_k(s)}v\| \sin \sphericalangle(\eta'(s), DR^k_{\eta_k(s)}v)}{\|\eta_k'(s)\| \cdot \|v\| \sin \sphericalangle(\eta_k'(s), v)} \\ \ge \frac{\|DR^k_{\eta_k(s)}\eta_k'(s)\|}{\|\eta_k'(s)\|} \cdot K\sigma^k \cdot \sin\theta$$

and so $\|\eta'(s)\| = \|DR^k_{\eta_k(s)}\eta_k'(s)\| \le \frac{C}{K \sin\theta}\left(\frac{\lambda}{\sigma}\right)^k \|\eta_k'(s)\|$ is uniformly forward contracted.

A stable backward invariant cone field

The length of η_k grows exponentially with k and, **since** η_k **is a** $\mathcal{C}^s$**-curve**, then η_k **crosses** N **transversely to the unstable cone field**.

In particular, $\mathcal{C}^s, \mathcal{C}^u$ behave as hyperbolic cone fields

- besides forward invariance of $\mathcal{C}^U$ we have $DR_x^{-1}\mathcal{C}^s_x \subset \mathcal{C}^s_{R^{-1}x}, x \in R(N)$;
- from the previous estimates we get
  - backward expansion: $\|DR_x^{-k} \cdot u\| \ge \frac{K\sin\theta}{C}\left(\frac{\sigma}{\lambda}\right)^k \|u\|$ for all $k \ge 1, u \in \mathcal{C}^s_x$ and $x \in R^k(N \setminus \Gamma)$;
  - domination: $\frac{\|DR^k_x v\|}{\|v\|} \ge K\sigma^k \frac{\|DR^k_x u\|}{\|u\|}$ for all $k \ge 1$, for all non-zero vectors $v \in \mathcal{C}^u_x, u \in DR_x^{-k} \cdot \mathcal{C}^s_{R^k x}$ and $x \in N$ such that $R^i x \notin \Gamma$ for $i = 0, \ldots, k-1$.

Conclusion in Case A

Hence, letting W be a smaller neighborhood if needed, we may assume without loss of generality that η_ℓ crosses ζ transversely in a single point $\{z_\ell\} = \eta_\ell \pitchfork \zeta$ (observe that η_ℓ cannot "bend" in N since it is tangent to the cone field $\mathcal{C}^s$).

Finally note that $R^\ell z_k \in W \cap R^{n+\ell} V$ and we have completed the proof of the transitivity Lemma in this case (Case A).

Now for the final case.

Case B *There exists* $y' \in B_\varepsilon(y), k \geq 1$ *and* $y'_k \in \Lambda_N$ *such that* $R^k y'_k = y'$ *and* $y'_k \in \Gamma^k_\varepsilon$.

The final Case B

Since $\Gamma^k_\varepsilon \subset \Gamma^1_\varepsilon$, we can find $x' \in V_0 \cap \xi$ such that $R^n x' \in \Gamma^k_\varepsilon$.

Hence we obtain that

$$R^{n+k} x', R^k y'_k \in B_\varepsilon(R^{k-1} \omega^{\pm})$$

which means in particular that $R^{n+k} x' \in B_{2\varepsilon}(y')$.

By the choice of ε, we see that $R^{n+k} x' \in B_{3\varepsilon}(y) \subset W$ and so $W \cap R^{n+k} V \neq \emptyset$.

This concludes the proof of the transitivity Lemma also in this case.