LECTURES ON SRB MEASURES

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1. SRB measures for hyperbolic attractors

1.1. **Topological attractor.** M - a compact smooth Riemannian manifold, $f : M \to M$ a C^2 (or $C^{1+\alpha}$) diffeomorphism.

$U \subset M$ open and $\overline{f(U)} \subset U$ - a *trapping region.*

$\Lambda = \bigcap_{n\geq 0} f^n(U)$ a *topological attractor* for f. We allow the case $\Lambda = M$.

Exercise 1. Show that Λ is compact, f-invariant and maximal (i.e., if $\Lambda' \subset U$ is invariant, then $\Lambda' \subset \Lambda$).

1.2. **Natural measures.** m is volume, $m_U = \frac{1}{m(U)} m|_U$ is the normalized volume in U,

$$\mu_n = \frac{1}{n}\sum_{k=0}^{n-1} f_*^k m_U \tag{1.1}$$

is an *evolution* of m.

Exercise 2. Show that the sequence μ_n is compact in the weak* topology.

There is $\mu_{n_k} \to \mu$ a *natural measure* for f on Λ. If and only if for any $h \in C^1(M)$:

$$\int_\Lambda h(x)d\mu_n = \frac{1}{n}\sum_{k=0}^{n-1}\int_\Lambda h(f^k(x))dm_U \to \int_\Lambda hd\mu.$$

Exercise 3. Show that μ is supported on Λ and is f-invariant.

1.3. **Basin of attraction.** μ is an (ergodic) measure on Λ.

$$B_\mu = \Big\{x \in U : \frac{1}{n}\sum_{k=0}^{n-1} h(f^k(x)) \to \int_\Lambda hd\mu \text{ for any } h \in C^1(M)\Big\}$$

basin of attraction of μ. We say that μ has *positive basin of attraction* if $\mu(B_\mu) > 0$.

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A (ergodic) natural measure μ on the attractor Λ is a *physical measure* if its basin of attraction has positive volume. An attractor with a physical measure is called a *Milnor attractor*.

1.4. Hyperbolic measures.

$$\chi(x,v) = \limsup_{n\to\infty} \frac{1}{n} \log \|df^n v\|, \quad x \in M.v \in T_xM$$

the *Lyapunov exponent* of v at x.

$\chi(x,\cdot)$ takes on finitely many values, $\chi_1(x) \le \cdots \le \chi_{p(x)}(x)$, $p(x) \le \dim M$ and $\chi(f(x)) = \chi(x)$, i.e., the values of the Lyapunov exponent are invariant functions, also $p(f(x)) = p(x)$.

μ is *hyperbolic* if $\chi_i(x) \ne 0$ and $\chi_1(x) < 0 < \chi_{p(x)}(x)$ that is

$$\chi_1(x) \le \dots \chi_k(x) < 0 < \chi_{k+1}(x) \le \cdots \le \chi_{p(x)}(x).$$

If μ is ergodic, then $\chi_i(x) = \chi_i(\mu)$ and $p(x) = p(\mu)$ for a.e. x that is

$$\chi_1(\mu) \le \dots \chi_k(\mu) < 0 < \chi_{k+1}(\mu) \le \cdots \le \chi_p(\mu).$$

If μ is hyperbolic, then for a.e. $x \in \Lambda$

(1) $T_xM = E^s(x) \oplus E^u(x)$ where

$$E^s(x) = \{v \in T_xM : \chi(x,v) < 0\}, \quad E^s(x) = \{v \in T_xM : \chi(x,v) < 0\}$$

stable and *unstable subspaces* at x and
   - (a) $dfE^{s,u}(x) = E^{s,u}(f(x))$,
   - (b) $\angle(E^s(x), E^u(x)) \ge K(x)$;

(2) there are $V^s(x)$, $V^u(x)$ *stable* and *unstable local manifolds* at x:
   - (a) we have

$$d(f^n(x), f^n(y)) \le C(x)\lambda^n(x) d(x,y), \quad y \in V^s(x),\ n \ge 0,$$
$$d(f^{-n}(x), f^{-n}(y)) \le C(x)\lambda^n(x) d(x,y), \quad y \in V^u(x),\ n \ge 0$$

   - (b) $C(x) > 0$, $K(x) > 0$,

$$C(f(x)) \le C(x)e^{\varepsilon(x)}, \quad K(f(x)) \ge K(x)e^{-\varepsilon(x)},$$

$0 < \lambda(x) < 1$, $\varepsilon(x) > 0$ and

$$\lambda(f(x)) = \lambda(x), \quad \varepsilon(f(x)) = \varepsilon(x).$$

(3) $r(x)$ the *size* of local manifolds and $r(f(x)) \ge r(x)e^{-\varepsilon(x)}$.

Exercise 4. Show that $V^u(x) \subset \Lambda$ for every $x \in \Lambda$ (for which the local unstable manifold is defined.

Fix $0 < \lambda < 1$ and set $\Lambda_\lambda = \{x \in \Lambda : 0 < \lambda(x) < \lambda\}$. Λ_λ is invariant and there is λ s.t. $\mu(\Lambda_\lambda) > 0$. We set $\Lambda = \Lambda_\lambda$.

$\ell > 1$, $\Lambda_\ell = \{x \in \Lambda : C(x) \le \ell,\ K(x) \ge \frac{1}{\ell}\}$ *regular set* of level ℓ.

(1) $\Lambda_\ell \subset \Lambda_{\ell+1}$, $\bigcup_{\ell\ge 1} \Lambda_\ell = \Lambda$;
(2) the subspaces $E^{s,u}(x)$ depend continuously on $x \in \Lambda$; in fact, Hölder continuously:
$$d_G(E^{s,u}(x), E^{s,u}(y)) \le M_\ell d(x,y)^\alpha,$$
where d_G is the Grasmannian distance in TM;
(3) the local manifolds $V^{s,u}(x)$ depend continuously on $x \in \Lambda$; in fact, Hölder continuously:
$$d_{C^1}(V^{s,u}(x), V^{s,u}(y)) \le L_\ell d(x,y)^\alpha;$$
(4) $r(x) \ge r_\ell$ for all $x \in \Lambda_\ell$.

1.5. **SRB measures.** We can assume that Λ_ℓ are compact and we can choose ℓ s.t. $\mu(\Lambda_\ell) > 0$. For $x \in \Lambda_\ell$ and small $\delta_\ell > 0$ set
$$Q_\ell(x) = \bigcup_{y \in B(x,\delta_\ell)\cap\Lambda_\ell} V^u(y).$$
Let ξ_ℓ be the partition of $Q_\ell(x)$ by $V^u(y)$, $\mu^u(y)$ the conditional measures generated by μ on $V^u(y)$ and $m_{V^u(y)}$ the leaf volume on $V^u(y)$. Both measures are probability (normalized) measures on $V^u(y)$.

μ is an *SRB measure* if μ is hyperbolic and for every ℓ with $\mu(\Lambda_\ell) > 0$, a.e. $x \in \Lambda_\ell$ and a.e. $y \in B(x,\delta_\ell) \cap \Lambda_\ell$, we have $\mu^u(y) \sim m_{V^u(y)}$.

For $y \in \Lambda_\ell$, $z \in V^u(y)$ and $n > 0$ set
$$\rho_n^u(y,z) = \prod_{k=0}^{n-1} \frac{\mathrm{Jac}(df|E^u(f^{-k}(z)))}{\mathrm{Jac}(df|E^u(f^{-k}(y)))}.$$

Exercise 5. Show that

(1) for every $\varepsilon > 0$ there is $N > 0$ s.t. for every $n \ge N$
$$\max_{y\in\Lambda_\ell} \max_{z \in V^u(y)} |\rho_n^u(y,z) - \rho^u(y,z)| \le \varepsilon;$$
in particular,
$$\rho^u(y,z) = \lim_{n\to\infty} \rho_n^u(y,z) = \prod_{k=0}^{\infty} \frac{\mathrm{Jac}(df|E^u(f^{-k}(z)))}{\mathrm{Jac}(df|E^u(f^{-k}(y)))};$$
(2) $\rho^u(y,z)$ depends continuously on $y \in \Lambda_\ell$ and $z \in V^u(y)$;
(3) $\rho^u(y,z)\rho^u(z,w) = \rho^u(y,w)$.

If μ is an SRB measure, then
$$d\mu^u(y)(z) = \rho^u(y)^{-1}\rho^u(y,z) dm_{V^u(y)}(z),$$
where
$$\rho^u(y) = \int_{V^u(y)} \rho^u(y,z)\, dm_{V^u(y)}(z) \tag{1.2}$$

is the normalizing factor; in particular, $\rho^u(y)^{-1}\rho^u(y,z)$ is the density of the SRB measure.

1.6. **Ergodic properties of SRB measures.**

Theorem 1.1 (Ledrappier). *Let μ be an SRB measure on an attractor Λ for a $C^{1+\alpha}$ diffeomorphism. Then there are $A_n \subset \Lambda$, $n = 1, 2, \dots$ s.t.*

(1) *$f(A_n) = A_n$, $\bigcup_{n\geq 0} = \Lambda$, $\mu(A_n) > 0$ for $n > 0$ and $\mu(A_0) = 0$;*
(2) *$f|A_n$ is ergodic;*
(3) *for each $n > 0$ there are $m_n \geq 1$ and $B_n \subset A_n$ s.t. the sets $f^i(B_n)$ are disjoint for $i = 0, \dots, m_n - 1$ and $f^{m_n}(B_n) = B_n$, $f^{m_n}|B_n$ is a Bernoulli diffeomorphism;*
(4) *for each $n > 1$ there are ℓ_n and $x_n \in \Lambda_{\ell_n}$ s.t.*
$$A_n = \bigcup_{m\in\mathbb{Z}} f^m(Q_{\ell_n}(x_n));$$

In addition, one can show that a hyperbolic measure μ on Λ is an SRB measure if and only if

(1) $\mu(B_\mu) = 1$;
(2) the Kolmogorov-Sinai entropy $h_\mu(f)$ of μ is given by the *entropy formula*:
$$h_\mu(f) = \int_\Lambda \sum_{\chi_i(x)>0} \chi_i(x)\, d\mu(x);$$

For smooth measures (which are a particular case of SRB measures) the upper bound for the entropy was obtained by Margulis (and extension to arbitrary Borel measures by Ruelle) and the lower bound was proved by Pesin (thus implying the entropy formula in this case). The extension to SRB measures was given by Ledrappier and Strelcyn. The fact that a hyperbolic measure satisfying the entropy formula is an SRB measure was proved by Ledrappier and for arbitrary (not necessarily hyperbolic) measures this result was obtained by Ledrappier and Young.

The limit measures for the sequence of measures (1.1) are natural candidates for SRB measures.

Exercise 6. Construct f s.t. $\mu_n \to \mu$, $\mu(B_\mu) > 0$ and

(1) μ is not hyperbolic;
(2) μ is hyperbolic but is not an SRB measure.

In this regard we state the following result by Tsujii.

Theorem 1.2. *Let Λ be an attractor for a $C^{1+\alpha}$ diffeomorphism f and suppose there is a positive Lebesgue measure set $S \subset \Lambda$ such that for every $x \in S$*

(1) *the sequence of measures $\frac{1}{n}\sum_{k=0}^{n-1} \delta_x$ converges weakly to an ergodic measure which we denote by μ_x;*
(2) *the Lyapunov exponents at x coincide with the Lyapunov exponents of the measure μ_x;*

(3) the measure μ_x has no zero and at least one positive Lyapunov exponent.

Then μ_x is an SRB measure for Lebesgue almost every $x \in S$.

It follows from Theorem 1.1 that f admits at most countably many ergodic SRB measures. J. Rodriguez Hertz, F. Rodriguez Hertz, R. Ures and A. Tahzibi have shown that a topologically transitive $C^{1+\alpha}$ surface diffeomorphism can have at most one SRB measure but the result is not true in dimension higher than two, see Section 2.4.

1.7. **Hyperbolic attractors.**

1.7.1. *Definition of hyperbolic attractors.* Λ a topological attractor for f. It is (uniformly) hyperbolic if for each $x \in \Lambda$ there is a decomposition of the tangent space $T_xM = E^s(x) \oplus E^u(x)$ and constants $c > 0$, $\lambda \in (0,1)$ s.t. for each $x \in \Lambda$:

(1) $\|d_xf^n v\| \leq c\lambda^n \|v\|$ for $v \in E^s(x)$ and $n \geq 0$;
(2) $\|d_xf^{-n} v\| \leq c\lambda^n \|v\|$ for $v \in E^u(x)$ and $n \geq 0$.

$E^s(x)$ and $E^u(x)$ are *stable* and *unstable subspaces* at x.

Exercise 7. Show that $E^s(x)$ and $E^u(x)$ depend continuously on x.

In particular, $\angle(E^s(x), E^u(x))$ is uniformly away from zero. In fact, $E^s(x)$ and $E^u(x)$ depend Hölder continuously on x.

For each $x \in \Lambda$ there are $V^s(x)$ and $V^u(x)$ *stable* and *unstable local manifolds* at x. They have uniform size r, depend continuously on x in the C^1 topology and $V^u(x) \subset \Lambda$ for any $x \in \Lambda$.

1.7.2. *An example of hyperbolic attractor.* Consider the solid torus $P = D^2 \times S^1$. We use coordinates (x, y, θ) on P; x and y give the coordinates on the disc, and θ is the angular coordinate on the circle. Fixing parameters $a \in (0,1)$ and $\alpha, \beta \in (0, \min\{a, 1-a\})$, define a map $f : P \to P$ by

$$f(x, y, \theta) = (\alpha x + a\cos\theta, \beta y + a\sin\theta, 2\theta).$$

The action of f on P may be described as follows:

(1) Take the torus and slice it along a disc so that it becomes a tube.
(2) Squeeze this tube so that its cross-sections are no longer circles of radius 1, but ellipses with axes of length α and β.
(3) Stretch the tube along its axis until it is twice its original length.
(4) Wrap the resulting longer, skinnier tube twice around the z-axis within the original solid torus.
(5) Glue the ends of the tube together.

P is a trapping region and $\Lambda = \bigcap_{n\geq 0} f^n(P)$ is the attractor for f, known as the Smale-Williams solenoid.

1.7.3. *Existence of SRB measures.*

Theorem 1.3 (Sinai, Ruelle, Bowen)**.** *Assume that f is C^2 (or $C^{1+\alpha}$). The following statements hold:*

(1) Every limit measure μ of the sequence of measures μ_n is an SRB measure on Λ.
(2) There are at most finally many ergodic SRB measures on Λ.
(3) If $f|\Lambda$ is topologically transitive, then the sequence of measures μ_n converges to a unique SRB-measure μ on Λ and B_μ has full measure in U.

Let μ be an SRB measure on Λ and A its ergodic component of positive measure. Then there is $x \in \Lambda$ s.t. $A = \bigcup_{m\in\mathbb{Z}} f^m(Q(x))$, where $Q(x) = \bigcup_{y\in V^u(x)} V^s(y)$. Note that $Q(x)$ is open and contains a ball of radius $\delta > 0$. This implies that μ has only finitely many ergodic components and they are open $\pmod 0$. It follows that there are at most finally many ergodic SRB measures on Λ and if $f|\Lambda$ is topologically transitive, then the SRB measure is unique.

To prove existence consider $x \in \Lambda$ and $V = V^u(x)$. For $y \in V^u(x)$ let

$$c_0 = 1 \text{ and } c_n = \Big(\prod_{k=0}^{n-1} \mathrm{Jac}(df|E^u(f^k(x)))\Big)^{-1} \text{ for } n \geq 1$$

and consider the sequence of measures given by

$$d\kappa_n(x)(y) = c_n\rho^u(f^n(x), y) dm_{f^n(V^u(x))}(y).$$

Lemma 1.4. $\kappa_n(x) = f^n_*\kappa_0(x)$.

Proof of the lemma. For a measurable set $F \subset V^u(x)$, $w \in F$ and $y = f^n(w) \in f^n(F)$,

$$\begin{aligned}
\kappa_n(F) &= \kappa_n(f^n(f^{-n}(F)) \\
&= \int_{f^{-n}(F)} c_n\rho^u(f^n(x), f^n(w)) \prod_{k=0}^{n-1} \mathrm{Jac}(df|E^u(f^k(w))\, dm_{V^u(x)}(w) \\
&= \int_{f^{-n}(F)} c_n\rho^u(f^n(x), f^n(w)) \prod_{k=0}^{n-1} \mathrm{Jac}(df|E^u(f^k(w))\, \rho(x,w)^{-1} d\kappa_0(x)(w) \\
&= \int_{f^{-n}(F)} d\kappa_0(x)(w) = \kappa_0(f^{-n}(F)).
\end{aligned}$$

Let

$$\nu_n = \frac{1}{n}\sum_{k=0}^{n-1} \kappa_k = \frac{1}{n}\sum_{k=0}^{n-1} f^k_*\kappa_0,$$

where we view κ_k and ν_n as measures on Λ. We shall show that every limit measure for this sequence of measures is an SRB-measure. In fact, every

SRB-measure can be constructed in this way, i.e., it can be obtained as the limit measure for a subsequence of measures ν_n.

We have that

$$d\nu_n(x)(y) = c_n\rho^u(f^n(x), y)m_n^u(x)(y),$$

where

$$m_n^u(x) = \frac{1}{n}\sum_{k=0}^{n-1} m_{f^k(V^u(x))}.$$

To prove that ν is an SRB measure. Let z be a Lebesgue point of ν, so that $\nu(B(z,r)) > 0$ for every $r > 0$. Consider the set $Q(z)$ and its partition ξ into unstable local manifolds $V^u(y)$, $y \in B(z,r)\cap\Lambda$. We identify the factor space $Q(z)/\xi$ with $W = V^s(z)\cap\Lambda$ and we denote by $V_n = f^n(V)$. Set

$$\begin{aligned}
&A_n = \{y \in W\colon V^u(y)\cap V_n \neq \emptyset\},\\
&B_n = \{y \in W\colon V^u(y)\cap \partial V_n \neq \emptyset\},\\
&C_n = A_n \setminus B_n, \quad D_n = \bigcup_{y\in B_n} V^u(y),\\
&F_n = \{y \in V_n : d^u(y, \partial V_n) \leq 2r\},
\end{aligned}$$

where d^u is the distance in V_n induced by the Riemannian metric. Note that $B_n \subset A_n$, $D_n \subset F_n$ and that A_n, B_n and C_n are finite set. If h is a continuous function on Λ with support in $Q(z)$, then

$$\begin{aligned}
\int_\Lambda h\,d\nu_n &= \int_{Q(z)} h\,d\nu_n\\
&= \sum_{y\in A_n}\int_{V^u(z)\cap V_n} h\,d\nu_n\\
&= \sum_{y\in C_n}\int_{V^u(z)\cap V_n} h\,d\nu_n + \sum_{y\in B_n}\int_{V^u(z)\cap V_n} h\,d\nu_n\\
&= I_n^{(1)} + I_n^{(2)}.
\end{aligned}$$

We have that

$$\begin{aligned}
\kappa_n(F_n) &= c_n\int_{F_n} \rho^u(f^n(x), y)\,dm_{f^n(V^u(x))}(y)\\
&= c_n\int_{f^{-n}(F_n)} \rho^u(f^n(x), f^n(w))\prod_{k=0}^{n-1}\mathrm{Jac}(df|E^u(f^k(w)))\Big)\, m_{V^u(x)}(w)\\
&= \int_{f^{-n}(F_n)} \rho^u(x,w)\,dm_{V^u(x)}(w) \leq C\kappa_0(f^{-n}(F_n)) \leq C(\lambda+\varepsilon)^{-n}2r,
\end{aligned}$$

where $C > 0$ is a constant and ε is sufficiently small so that $\lambda + \varepsilon < 1$. It follows that

$$I_n^{(2)} \leq C\nu_n(D_n) \leq C\nu_n(F_n) \leq \frac{C}{n}.$$

Denote by δ_n the measure on W, which is the uniformly distributed point mass on C_n. We have that

$$\int_\Lambda h\, d\kappa_n = c_n \sum_{y\in C_n} \rho^u(f^n(x),y) \int_{V^u(y)} h(w)\rho^u(y,w)\, dm_{V^u(y)}(w)$$

$$= \int_W c_n\rho^u(f^n(x),y)\rho^u(y)\, d\delta_n(y) \int_{V^u(y)} h(w)\frac{\rho^u(y,w)}{\rho^u(y)}\, dm_{V^u(y)}(w),$$

where $\rho^u(y)$ is given by (1.2). Hence,

$$I_n^{(2)} = \frac{1}{n}\sum_{k=0}^{n-1} \int_W c_k\rho^u(f^k(x),y)\rho^u(y)\, d\delta_k(y) \int_{V^u(y)} h(w)\frac{\rho^u(y,w)}{\rho^u(y)}\, dm_{V^u(y)}(w),$$

The desired result is now a corollary of the following statement.

Lemma 1.5. *Let ν_n be a sequence of Borel probability measures on $Q(z)$ such that*

(1) if $(\delta_n, \nu_n^u(y))$ is the systems of conditional measures for ν_n with respect to the partition ξ, so that δ_n is a measure on the factor space $W = Q(z)/\xi$ and $\nu_n^u(y))$ is a measure on $V^u(y)$, then

$$d\nu_n^u(y)(w) = P_n(y,w)dm_{V^u(y)}(w),$$

where $P_n(y,w)$ is a continuous function on $Q(z)$;

(2) there is a sequence of numbers n_ℓ s.t. the sequence of measures ν_{n_ℓ} converges in the weak topology to a measure ν on $Q(z)$;*

(3) the sequence of functions $P_{n_\ell}(y,w)$ converges uniformly in $Q(z)$ to a continuous function $P(y,w)$.

Then the system of conditional measures for ν with respect to the partition ξ has the form $(\delta, \nu^u(y))$ where δ is the measure on the factor space W that is the limit of measures δ_{n_ℓ} and $\nu^u(y)$ is a measure on $V^u(y)$ for which

$$d\nu^u(y)(w) = P(y,w)dm_{V^u(y)}(w).$$

We stress that in the definition of the sequence of measures ν_n one can replace the local unstable manifold $V^u(x)$ with any *admissible* manifold, i.e., a local manifold passing through x and sufficiently close to $V^u(x)$ in the C^1 topology.

If $f|\Lambda$ is topologically transitive, then the SRB measure is unique and hence, the sequence of measures converges to ν.

Exercise 9. Show that the sequence of measures μ_n converges to ν.

In the particular, case when $\Lambda = M$ that is f is a C^2 Anosov diffeomorphism, the above theorem guaranties existence and uniqueness of the SRB measure μ for f (provided f is topologically transitive). Reversing the time we obtain the unique SRB measure ν for f^{-1}. One can show that $\mu = \nu$ if and only if μ is a smooth measure.

2. SRB measures for partially hyperbolic attractors

2.1. **Definition of partially hyperbolic attractors.** Λ a topological attractor for f. It is (uniformly) partially hyperbolic if for each $x \in \Lambda$ there is a decomposition of the tangent space $T_xM = E^s(x) \oplus E^c(x) \oplus E^u(x)$ and for $n \geq 0$:

(1) $\|d_xf^n v\| \leq c\lambda^n \|v\|$ for $v \in E^s(x)$;
(2) $c^{-1}\lambda_1^n \|d_xf^n v\| \leq c\lambda_2^n \|v\|$;
(3) $\|d_xf^{-n} v\| \leq c\lambda^n \|v\|$ for $v \in E^u(x)$.

$E^s(x)$, $E^c(x)$ and $E^u(x)$ are *strongly stable*, *central* and *strongly unstable subspaces* at x. They depend (Hölder) continuously on x. In particular, the angle between any two of them is uniformly away from zero.

For each $x \in \Lambda$ there are $V^s(x)$ and $V^u(x)$ *strongly stable* and *strongly unstable local manifolds* at x. They have uniform size r, depend continuously on x in the C^1 topology and $V^u(x) \subset \Lambda$ for any $x \in \Lambda$.

An example of a hyperbolic attractor is a map which the direct product of a map f with a hyperbolic attractor Λ and Id map of any manifold.

2.2. ***u*-measures.** μ is a *u-measure* if for every $x \in \Lambda$ and $y \in B(x,\delta) \cap \Lambda$, we have $\mu^u(y) \sim m_{V^u(y)}$.

Theorem 2.1 (Pesin, Sinai). *Any limit measure of the sequence of measures μ_n is a u-measure and so is any limit measure of the sequence of measures ν_n.*

Properties of *u*-measures:

(1) Any measure whose basin has positive volume is a u-measure;
(2) If there is a unique u-measure for f, then its basin has full volume in the topological basin of attraction;
(3) Every ergodic component of a u-measure is again a u-measure.

Exercise 10. Give an example of a map f with a partially hyperbolic attractor and a u-measure whose basin has zero volume.

2.3. ***u*-measures with negative central exponents.** We say that f has *negative (positive) central exponents* (with respect to μ) if there exists an invariant subset $A \subset \Lambda$ with $\mu(A) > 0$ s.t. the Lyapunov exponents $\chi(x,v) < 0$ (respectively, $\chi(x,v) > 0$) for every $x \in A$ and every vector $v \in E^c(x)$.

If f has negative central exponents on a set A of full measure with respect to a u-measure μ, then μ is an SRB measure for f.

Theorem 2.2. *Assume that f has negative central exponents on an invariant set A of positive measure with respect to a u-measure μ for f. Then the following statements hold:*

(1) *Every ergodic component of $f|A$ of positive μ-measure is open* (mod 0)*; in particular, the set A is open* (mod 0) *(that is there exists an open set U s.t. $\mu(A \triangle U) = 0$).*
(2) *If for μ-almost every x the trajectory $\{f^n(x)\}$ is dense in supp(μ), then f is ergodic with respect to μ.*

Proof. $f|A$ has non-zero Lyapunov exponents with $T_xM = E^-(x) \oplus E^u(x)$ for every $z \in A$ where $E^-(x) = E^s(x) \oplus E^c(x)$. While the unstable local manifold $V^u(x)$ have size which is uniformly away from zero, the stable local manifolds $V^-(x)$ have variable sizes and we will consider the collection of regular sets A_ℓ. The measure $\mu|A$ is an SRB measure for $f|A$ and hence, it has at most countably many ergodic components of positive measure. Each such component is of the form $\bigcup_{n\in\mathbb{Z}} f^n(Q_\ell(x))$ where

$$Q_\ell(x) = \bigcup_{y\in B(x,\delta_\ell)\cap A_\ell} V^u(y).$$

This set is open (mod 0) and so is every ergodic component and hence the set A itself. The first statement follows. Under the second assumption $f|A$ is topologically transitive and hence, $F|A$ is ergodic. It remains to show that $A = \Lambda$ (0). Assume for a contradiction that $D = \Lambda \setminus A$ has nonzero measure. In particular, $D \subset \text{supp}(\mu)$. Since A is open (mod 0), it follows from the assumption that almost every trajectory is dense, that we can choose $n \geq 1$ such that $\mu\{x \in D : f^n(x) \in A\} > 0$. However, this contradicts the f-invariance of A (and of D).

We provide the following criterion, which guarantees the density assumption in Statement (2) of the previous theorem.

Theorem 2.3. *Assume that for every $x \in \Lambda$ the orbit of the global strongly unstable manifold $W^u(x)$ is dense in Λ. Then for any u-measure μ on Λ and μ-almost every x the trajectory $\{f^n(x)\}$ is dense in Λ.*

This result is an immediate corollary of the following more general statement. Given $\varepsilon > 0$, we say that a set is *ε-dense* if its intersection with any ball of radius ε is not empty.

Theorem 2.4. *Let f be a C^1 diffeomorphism of a compact smooth Riemannian manifold M possessing a partially hyperbolic attractor Λ. The following statements hold:*

(1) *For every $\delta > 0$ and every $\varepsilon \leq \delta$ the following holds: assume that for every $x \in \Lambda$ the orbit of the global strongly unstable manifold $W^u(x)$ is ε-dense in Λ. Then for any u-measure μ on Λ and μ-almost every x the trajectory $\{f^n(x)\}$ is δ-dense in Λ.*
(2) *Assume that for every $x \in \Lambda$ the orbit of the global strongly unstable manifold $W^u(x)$ is dense in Λ. Then supp(μ) $= \Lambda$ for every u-measure μ.*

Proof. The second statement is an immediate corollary of the first statement. To prove the first statement choose an open set $U \subset \Lambda$. There is a ball $B(\delta)$ of radius δ that is contained in U. It follows from the assumption of Statement 1 that for every $x \in \Lambda$ there exists $n = n(x, U)$ s.t. $f^n(W^u(x)) \cap U \neq \emptyset$. Let now μ be a u-measure on Λ. We shall show that μ-almost every $x \in \Lambda$ there is $m = m(x)$ s.t. $f^m(x) \in U$. To this end consider the set Y of points whose positive semi-trajectories never visit U and assume by contradiction that Y has positive μ-measure. Then by the definition of u-measure, there is a point $x \in \Lambda$ s.t. $m^u(V^u(x) \cap Y) > 0$. It follows that there is a point $y \in V^u(x) \cap Y$ s.t. for every $\gamma > 0$ one can find $r > 0$ such that

$$\frac{m^u(B^u(y,r) \cap Y)}{m^u(B^u(x,r))} \geq 1 - \gamma,$$

where $B^u(y,r)$ is an r-ball in the leaf $V^u(x)$ centered at y. However, this contradict the following statement.

Lemma 2.5. *There exists $\eta > 0$ s.t. for every $x \in \Lambda$ and every $r > 0$*

$$\frac{m^u(B^u(x,r) \setminus Y)}{m^u(B^u(x,r))} > \eta,$$

where $B^u(x,\delta)$ is a δ-ball in the leaf $V^u(x)$ centered at x and m^u is the leaf volume in $V^u(x)$.

Proof of the lemma. Given $x \in \Lambda$ and $\delta > 0$, set $A_{x,\delta} = B^u(x,\delta) \setminus Y$. Observe that there is $\gamma > 0$ such that for all x,

$$\frac{m^u(B^u(x,\delta))}{m^u(B^u(x,\delta(1+\gamma)))} \geq \frac{1}{2}.$$

Given $\Delta > 0$, we can choose $m \geq 1$ s.t. for all $y \in \Lambda$,

$$f^m(B^u(y, \delta\gamma/2)) \supset B^u(f^m(y), \Delta). \tag{2.1}$$

We can then choose a cover by Δ-balls,

$$f^m(B^u(x,\delta)) \subset \bigcup_i B^u(f^m(x_i), \Delta).$$

By (2.1), we obtain that

$$B^u(f^m(x_i), \Delta) \subset f^m(B^u(x, \delta(1+\gamma))).$$

In particular,

$$\frac{m^u(A_{x,\delta(1+\gamma)})}{m^u(B^u(x,\delta))} \geq \frac{m^u(\bigcup_i f^{-m}(A_{f^m(x_i),\Delta}))}{m^u(\bigcup_i f^{-m}(B^u(f^m(x_i),\Delta)))}.$$

Moreover, using the Besicovitch Covering Lemma, we can assume without loss of generality that for this cover each point lies in at most K balls, for some fixed constant $K > 0$. We then have a lower bound

$$\frac{m^u(\bigcup_i f^{-m}(A_{f^m(x_i),\Delta}))}{m^u(\bigcup_i f^{-m}(B^u(f^m(x_i),\Delta)))} \geq \frac{1}{K} \frac{\sum_i m^u(f^{-m}(A_{f^m(x_i),\Delta}))}{\sum_i m^u(f^{-m}(B^u(f^m(x_i),\Delta)))}.$$

Using standard bounded distortion estimates we can write

$$\frac{m^u(f^{-m}(A_{f^m(x_i),\Delta}))}{m^u(f^{-m}(B^u(f^m(x_i),\Delta)))} = \frac{\int_{A_{f^m(x_i),\Delta}} \mathrm{Jac}(df^{-m})dm^u}{\int_{B^u(f^m(x_i),\Delta)} \mathrm{Jac}(df^{-m})dm^u}$$

$$\geq c\frac{m^u(A_{x_i,\Delta})}{m^u(B^u(x_i,\Delta))},$$

where

$$c = \inf_{m\geq 0} \inf_{y_1,y_2\in B^u(x,\Delta)} \frac{\mathrm{Jac}(d_{y_1}f^{-m})}{\mathrm{Jac}(d_{y_2}f^{-m})} > 0$$

Observe that the set Y is compact and that the leaf volumes $m^u(y)$ vary continuously with $y \in Y$. It follows that we can choose $\Delta > 0$ such that

$$\rho = \min_{y\in\Lambda}\left\{\frac{m^u(A_{y,\Delta})}{m^u(B^u(y,\Delta))}\right\} > 0.$$

Combining all of the above inequalities we get

$$\frac{m^u(A_{x,\delta(1+\gamma)})}{m^u(B^u(x,\delta(1+\gamma)))} \geq \frac{c\rho}{2K}.$$

Since $\delta > 0$ can be chosen arbitrarily small, the proof of the lemma is complete.

2.4. **Uniqueness of u-measures and SRB measures.** In the case of a hyperbolic attractor, topological transitivity of $f|\Lambda$ guarantees that there is a unique u-measure for f on Λ. In contrast, in the partially hyperbolic situation, even topological mixing is not enough to guarantee that there is a unique u-measure. Indeed, consider $F = f_1 \times f_2$, where f_1 is a topologically transitive Anosov diffeomorphism and f_2 a diffeomorphism close to the identity. Then any measure $\mu = \mu_1 \times \mu_2$, where μ_1 is the unique SRB measure for f_1 and μ_2 any f_2-invariant measure, is a u-measure for F. Thus, F has a unique u-measure if and only if f_2 is uniquely ergodic. On the other hand, F is topologically mixing if and only if f_2 is topologically mixing.

Theorem 2.6. *Let f be a $C^{1+\alpha}$ diffeomorphism of a compact smooth Riemannian manifold M possessing a partially hyperbolic attractor Λ. Assume that:*

(1) *there exists a u-measure μ for f with respect to which f has negative central exponents on an invariant subset $A \subset \Lambda$ of positive μ-measure;*
(2) *for every $x \in \Lambda$ the orbit of the global strongly unstable manifold $W^u(x)$ is dense in Λ.*

Then μ is the only u-measure for f and f has negative central exponents at μ-almost every $x \in \Lambda$. In particular, (f,μ) is ergodic, supp$(\mu) = \Lambda$, and the basin $B(\mu)$ has full volume in the topological basin of attraction of Λ. μ is the only SRB measure for f.

Proof. Let μ be a u-measure for f with negative central exponents on a subset $A \subset \Lambda$ of positive measure. It follows from the previous theorem that f has negative central exponents μ-a.e. and is ergodic with respect to μ. Let now ν be a u-measure for f (we do not assume at this point that ν has negative central exponents on a set of positive ν-measure). By the assumptions of the theorem, for every $z \in \Lambda$ the intersection $W^u(f^n(z)) \cap V^-(y)$ is not empty for some $n \in \mathbb{Z}$ and for every $y \in V^u(x) \cap A_\ell$. Moreover, by the absolute continuity property of local stable manifolds, for every $z \in \Lambda$ the intersection $W^u(f^n(z)) \cap B$ has positive leaf volume where $B = \bigcap_{y \in A_\ell} V^-(y)$. Since ν is a u-measure, it follows that f has negative central exponents on an invariant subset $A_\nu \subset \Lambda$ of positive ν-measure. We conclude that f has negative central exponents ν-a.e. and is ergodic with respect to ν. Note that f has negative central exponents a.e. with respect to the measure $\frac{1}{2}(\mu + \nu)$ and is ergodic with respect to this measure. This implies that $\mu = \nu$.

2.5. **Small perturbations of systems with zero central exponents.** Shub and Wilkinson considered the direct product $F_0 = f \times \mathrm{Id}$, where f is a linear Anosov diffeomorphism and the identity acts on the circle. The map F_0 preserves volume. Shub and Wilkinson showed that arbitrary close to F_0 (in the C^1 topology) there is a volume-preserving diffeomorphism F whose only central exponent is negative on the whole of M.

2.6. **Density of unstable leaves.** Bonatti and Diaz have shown that there is an open set of transitive diffeomorphisms near $F_0 = f \times \mathrm{Id}$ (f is an Anosov diffeomorphism and Id is the identity map of any manifold) as well as near the time-1 map $F - 0$ of a topologically transitive Anosov flow. This result was used by Bonatti and Diaz and Ures to construct examples of partially hyperbolic systems with minimal unstable foliation (i.e., every unstable leaf is dense in the manifold itself).

If f is a small perturbation of F_0 then f is partially hyperbolic and by [**?**], the central distribution of f is integrable. Furthermore, the central leaves are compact in the first case and there are compact leaves in the second case.

Theorem 2.7. *Assume that there is a compact periodic central leaf C for f such that $f^n(C) = C$ and the restriction $f^n|C$ is a minimal transformation. Then the unstable foliation for f is minimal.*

2.7. **Stable ergodicity for dissipative maps.** Let Λ_f be a topological attractor for a diffeomorphism f. We say that f is *stably ergodic* if there exists a neighborhood $\mathcal{U}$ of f in $\mathrm{Diff}^r(M)$, $r \geq 1$ s.t. any diffeomorphism $g \in \mathcal{U}$ possesses a topological attractor Λ_g and there is a unique SRB measure μ_g on Λ_g (and hence, g is ergodic with respect to μ_g).

If the attractor Λ_f is (partially) hyperbolic then there exists a neighborhood $\mathcal{U}$ of f in $\mathrm{Diff}^1(M)$ s.t. any diffeomorphism $g \in \mathcal{U}$ possesses a (partially) hyperbolic attractor Λ_g.

Theorem 2.8. *Let Λ_f be a partially hyperbolic attractor for a diffeomorphism f. If f satisfies the conditions of Theorem 3.6, then f is stably ergodic with $r = 1 + \alpha$.*

Proof. f is ergodic with respect to its unique SRB measure. This measure is a unique u-measure with negative central exponents a.e. Therefore, there exists $a > 0$ s.t. for a.e. $x \in \Lambda_f$,

$$\lim_{n\to+\infty} \frac{1}{n} \ln \|df^n|E^c_f(x)\| < -a.$$

Integrating over Λ_f we obtain

$$\lim_{n\to\infty} \frac{1}{n} \int_{\Lambda_f} \ln \|df^n|E^c_f(x)\|\, d\mu(x) < -a.$$

In particular, there exists $n_0 > 0$ such that

$$\frac{1}{n_0} \int_{\Lambda_f} \ln \|df^{n_0}|E^c_f(x)\|\, d\mu(x) < -\frac{a}{2}.$$

Without loss of generality we may assume that $n_0 = 1$, so that

$$\int_{\Lambda_f} \ln \|df|E^c_f(x)\|\, d\mu(x) < -\frac{a}{2}.$$

If a diffeomorphism g is sufficiently close to f in the $C^{1+\alpha}$ topology, then for any u-measure ν on Λ_g we have

$$\int_{\Lambda_g} \ln \|dg|E^c_g(x)\|\, d\nu(x) < -\frac{a}{4}.$$

Take a u-measure μ_g for g on Λ_g. It follows that there exists a subset A_g with $\mu_g(A_g) > 0$ s.t. for every $x \in A_g$

$$\lim_{n\to+\infty} \frac{1}{n} \sum_{j=0}^{n-1} \ln \|dg|E^c_g(g^j(x))\| \le -\frac{a}{4}.$$

Hence,

$$\lim_{n\to+\infty} \frac{1}{n} \ln \|dg^n|E^c_g(x)\| \le -\frac{a}{4}$$

for every $x \in A_g$ and hence, μ_g has negative central exponents on a set of positive measure. Restricting to an ergodic component of positive measure, we may assume that μ_g is ergodic. We need the following result:

Lemma 2.9. *Let f be a $C^{1+\alpha}$ diffeomorphism possessing a partially hyperbolic attractor Λ_f. Then for every $a > 0$ there exist $r_0 = r_0(a, f) > 0$, which depends continuously on f in the $C^{1+\alpha}$ topology, s.t. the following statement holds. Let μ be a u-measure for f with negative central exponents on an invariant subset A of positive μ-measure. Assume that $|\chi(x, v)| \ge a$ for μ-a.e. $x \in A$ and all $v \in T_xM$. Then for μ-a.e. $x \in A$ there is $n - n(x) \ge 0$ s.t. the size $V^-(f^{-n}(x)$ is at least r_0.*

Take a small ε. If g is sufficiently close to f, then the orbit of $W^u(x)$ is ε-dense in Λ_g. It follows that any u-measure ν for g coincides with μ_g and the desired result follows.

One can show that in fact if f has a unique SRB measure μ_f with negative central exponents, then f is stably ergodic with $r = 1 + \alpha$.

Our results clearly hold true for uniformly hyperbolic (Axiom A) attractors (for which $E^c = 0$). In particular, our approach gives a proof of stable ergodicity of topologically transitive Axiom A attractors.

3. SRB measures for non-uniformly hyperbolic attractors

Problem. Assume that $\chi_i(x) \neq 0$ for every i and m_U-a.e. x. Then there is $\mu_{n_k} \to \mu$ and i s.t. $\chi_i(x) \neq 0$ for μ-a.e. x.

3.1. **Statements.** Given $x \in M$, a subspace $E(x) \subset T_xM$, and $\theta(x) > 0$, the *cone* at x around $E(x)$ with angle $\theta(x)$ is

$$K(x, E(x), \theta(x)) = \{v \in T_xM \mid \measuredangle(v, E(x)) < \theta(x)\}.$$

If E is a measurable distribution on $A \subset M$ and the angle function θ is measurable, then we have a *measurable cone family* on A.

We make the following standing assumption.

(H1) There exists a forward-invariant set $A \subset U$ of positive volume with two measurable cone families $K^s(x), K^u(x) \subset T_xM$ s.t.
   (a) $\overline{Df(K^u(x))} \subset K^u(f(x))$ for all $x \in A$;
   (b) $\overline{Df^{-1}(K^s(f(x)))} \subset K^s(x)$ for all $x \in f(A)$.
   (c) $K^s(x) = K(x, E^s(x), \theta_s(x))$ and $K^u(x) = K(x, E^u(x), \theta_u(x))$ are s.t. $T_xM = E^s(x) \oplus E^u(x)$; moreover $d_s = \dim E^s(x)$ and $d_u = \dim E^u(x)$ do not depend on x.

Such cone families automatically exist if f is uniformly hyperbolic on Λ. We emphasize, however, that in our setting $K^{s,u}$ are not assumed to be continuous, but only measurable and the families of subspaces $E^{u,s}(x)$ are not assumed to be invariant.

Let $A \subset U$ be a forward-invariant set satisfying (H1). Define

$$\lambda^u(x) = \inf\{\log \|Df(v)\| \mid v \in K^u(x), \|v\| = 1\},$$
$$\lambda^s(x) = \sup\{\log \|Df(v)\| \mid v \in K^s(x), \|v\| = 1\}.$$

We define the *defect from domination* at x to be

$$\Delta(x) = \tfrac{1}{\alpha} \max(0,\ \lambda^s(x) - \lambda^u(x)),$$

where $\alpha \in (0, 1]$ is the Hölder exponent of Df. Roughly speaking, $\Delta(x)$ controls how much the curvature of unstable manifolds can grow as we go from x to $f(x)$.

The following quantity is positive whenever f expands vectors in $K^u(x)$ and contracts vectors in $K^s(x)$:

$$\lambda(x) = \min(\lambda^u(x) - \Delta(x), -\lambda^s(x)).$$

The *upper asymptotic density* of $\Gamma \subset \mathbb{N}$ is

$$\overline{\delta}(\Gamma) = \limsup_{N\to\infty} \frac{1}{N} \#\Gamma \cap [0, N).$$

An analogous definition gives the lower asymptotic density $\underline{\delta}(\Gamma)$.

We say that a point $x \in A$ is *effectively hyperbolic* if

(EH1)

$$\liminf_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} \lambda(f^k(x)) > 0,$$

(EH2)

$$\lim_{\bar\theta\to 0} \overline{\delta}\{n \mid \theta(f^n(x)) < \bar\theta\} = 0.$$

Condition (EH1) says that not only are the Lyapunov exponents of x positive for vectors in K^u and negative for vectors in K^s, but λ^u gives enough expansion to overcome the 'defect from domination' given by Δ.

Condition (EH2) requires that the frequency with which the angle between the stable and unstable cones drops below a specified threshold $\bar\theta$ can be made arbitrarily small by taking the threshold to be small.

If Λ is a hyperbolic attractor for f, then **every** point $x \in U$ is effectively hyperbolic.

Let A satisfy (H1), and let $S \subset A$ be the set of effectively hyperbolic points. Observe that effective hyperbolicity is determined in terms of a forward asymptotic property of the orbit of x, and hence S is forward invariant under f.

Theorem 3.1. *Let f be a $C^{1+\alpha}$ diffeomorphism of a compact manifold M, and Λ a topological attractor for f. Assume that*

(1) f admits measurable invariant cone families as in (H1);
(2) the set S of effectively hyperbolic points satisfies $LebS > 0$.

Then f has an SRB measure supported on Λ.

A similar result can be formulated given information about the set of effectively hyperbolic points on a single 'approximately unstable' submanifold usually called *admissible*; the precise definition is not needed for the statement of the theorem; all we need here is to have $T_xW \subset K^u(x)$ for 'enough' points x. $W \subset U$. Let d_u, d_s, and A be as in (H1), (EH1) and (EH2), and let $W \subset U$ be an embedded submanifold of dimension d_u.

Theorem 3.2. *Let f be a $C^{1+\alpha}$ diffeomorphism of a compact manifold M, and Λ a topological attractor for f. Assume that*

(1) f admits measurable invariant cone families as in (H1);
(2) there is a d_u-dimensional embedded submanifold $W \subset U$ s.t. $m_W(\{x \in S \cap W \mid T_xW \subset K^u(x)\}) > 0$.

Then f has an SRB measure supported on Λ.

3.2. **Related results.** Let f be a C^2 diffeomorphism and A a forward-invariant compact set. A splitting $T_AM = E^s \oplus E^u$ is *dominated* if there is $\chi < 1$ s.t.

$$\|Df|_{E^s(x)}\| < \chi \|Df|^{-1}_{E^u(x)}\|^{-1} \text{ for all } x \in A;$$

equivalently, the splitting is dominated if $\lambda^s(x) < \lambda^u(x)$ for all $x \in A$. Alves, Bonatti, and Viana considered systems with a dominated splitting for which

(1) E^s is *uniformly contracting*: $\lambda^s(x) \le -\bar{\lambda} < 0$ for all $x \in A$;
(2) E^u is *mostly expanding*: there is $\tilde{S} \subset A$ with positive volume and

$$\liminf_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \lambda^u(f^j x) > 0 \text{ for all } x \in \tilde{S}. \tag{3.1}$$

Under these conditions they proved that f has an SRB measure supported on $\Lambda = \bigcap_{j=0}^{\infty} f^j(A)$, and that the same result is true if (4.1) holds on a positive Lebesgue measure subset of some disk transverse to E^s. A similar result for the (easier) case when E^u is uniformly expanding and E^s is mostly contracting was given earlier by Bonatti and Viana. A stronger version of this result was recently obtained by Alves, Diaz, Luzzatto and Pinheiro.

Given a dominated splitting with a uniformly contracting E^s, we see immediately that $\Delta(x) = 0$ and $\lambda(x) = \lambda^u(x)$ for all $x \in A$, so that our requirement is equivalent to (3.1). Moreover, by continuity and compactness, the angle between E^u and E^s is bounded away from 0, so our second requirement is automatic, and we conclude that the set $\tilde{S}$ in the above result is exactly the set S from our theorems.

The proof of the above result requires the notion of *hyperbolic times*, introduced by Alves. These are times n s.t. for some fixed $\sigma < 1$, and every $0 \le k \le n$, we have

$$\prod_{j=n-k+1}^{n} \|Df^{-1}|_{E^{cu}_{f^j(x)}}\| \le \sigma^k; \text{ equivalently, } \sum_{j=n-k}^{n-1} \lambda^u(f^j x) \ge k|\log\sigma|.$$

If x satisfies (3.1), then Pliss' lemma guarantees that the set of hyperbolic times for x has positive lower asymptotic density. A similar strategy runs through the heart of our main results: our conditions (EH1)–(EH2) guarantee a positive lower asymptotic density of *effective hyperbolic times* at which we can apply a version of the Hadamard–Perron theorem, allowing us to carry out the geometric construction of an SRB measure.

Overall, we can summarize the situation as follows. In the geometric approach to construction of SRB measures, one needs good information on the dynamics and geometry of admissible manifolds and their images. Ideally one wants **hyperbolicity**: the unstable direction expands, the stable direction contracts. If this happens all the time, we are in the uniformly hyperbolic setting and one can carry out the construction without too much trouble. If hyperbolicity does not hold all the time, then we are in the non-uniformly hyperbolic setting and need two further conditions in order to play the game.

(1) Domination: if one of the directions does not behave hyperbolically, then it at least is still dominated by the other direction.
(2) Separation: the stable and unstable directions do not get too close to each other.

In the case of dominated splittings these two conditions hold uniformly and so one only needs to control the asymptotic hyperbolicity (expansion and contraction along stable and unstable directions). For our more general setting, both domination and separation may fail at some points, and in order to control the geometry and dynamics of images of admissible manifolds, we need to replace 'hyperbolicity' with 'effective hyperbolicity'. The two conditions (EH1) and (EH2) control the failures of domination and separation, respectively: the presence of $\Delta(x)$ lets us control curvature of admissible manifolds when domination fails, and the condition on $\theta(x)$ guarantees that separation does not fail too often.

3.3. **Maps on the boundary of Axiom A: neutral fixed points.** We give a specific example of a map for which the conditions of our main theorem can be verified. Let $f\colon U \to M$ be a $C^{1+\alpha}$ Axiom A diffeomorphism onto its image with $\overline{f(U)} \subset U$, where $\alpha \in (0,1)$. Suppose that f has one-dimensional unstable bundle.

Let p be a fixed point for f. We perturb f to obtain a new map g that has an indifferent fixed point at p. The case when M is two-dimensional and f is volume-preserving was studied by Katok. We allow manifolds of arbitrary dimensions and (potentially) dissipative maps. For example, one can choose f to be the Smale-Williams solenoid or its sufficiently small perturbation.

We suppose that there exists a neighborhood $Z \ni p$ with local coordinates in which f is the time-1 map of the flow generated by

$$\dot{x} = Ax$$

for some $A \in GL(d, \mathbb{R})$. Assume that the local coordinates identify the splitting $E^u \oplus E^s$ with $\mathbb{R} \oplus \mathbb{R}^{d-1}$, so that $A = A_u \oplus A_s$, where $A_u = \gamma \mathrm{Id}_u$ and $A_s = -\beta \mathrm{Id}_s$ for some $\gamma, \beta > 0$. In the Katok example we have $d = 2$ and $\gamma = \beta$ since the map is area-preserving.

Now we use local coordinates on Z and identify p with 0. Fix $0 < r_0 < r_1$ s.t. $B(0, r_1) \subset Z$, and let $\psi\colon Z \to [0,1]$ be a $C^{1+\alpha}$ function such that

(1) $\psi(x) = \|x\|^\alpha$ for $\|x\| \leq r_0$;
(2) $\psi(x) = 1$ for $\|x\| \geq r_1$;
(3) $\psi(x) > 0$ for $x \neq 0$ and $\psi'(x) > 0$.

Let $\mathcal{X}\colon Z \to \mathbb{R}^d$ be the vector field given by $\mathcal{X}(x) = \psi(x)Ax$. Let $g\colon U \to M$ be given by the time-1 map of this vector field on Z and by f on $U \setminus Z$. Note that g is $C^{1+\alpha}$ because $\mathcal{X}$ is $C^{1+\alpha}$.

Theorem 3.3. *The map g has an SRB measure.*

Note that g does not have a dominated splitting because of the indifferent fixed point. We also observe that if ψ is taken to be C^∞ away from 0, then g is also C^∞ away from the point p.

3.4. **Outline of the proof.**

3.4.1. *Description of geometric approach for uniformly hyperbolic attractors.* We revisit the construction of SRB measures for uniformly hyperbolic attractors Λ for f. Note that in this case the cones $K^u(x)$ and $K^s(x)$ can be extended to the neighborhood U and are continuous. Let $W \subset U$ be an *admissible manifold*; that is, a d_u-dimensional submanifold that is tangent to an unstable cone $K^u(x)$ at some point $x \in U$ and has a fixed size and uniformly bounded curvature.

Consider the leaf volume m_W on W and take the pushforwards $f_*^n m_W$ given by

$$(f_*^n m_W)(E) = m_W(f^{-n}(E)). \tag{3.2}$$

To obtain an invariant measure, we take Césaro averages:

$$\mu_n = \frac{1}{n}\sum_{k=0}^{n-1} f_*^k m_W. \tag{3.3}$$

By weak* compactness there is a subsequence μ_{n_k} that converges to an invariant measure μ on Λ which is an SRB measure. We present an argument that can be adapted to our setting of effective hyperbolicity.

Consider the images $f^n(W)$ and observe that for each n, the measure $f_*^n m_W$ is absolutely continuous with respect to leaf volume on $f^n(W)$. For every n, the image $f^n(W)$ can be covered with uniformly bounded multiplicity (this requires a version of the Besicovitch covering lemma) by a finite number of admissible manifolds W_i, so that

$$f_*^n m_W \text{ is a convex combination of measures } \rho_i \, dm_{W_i}, \tag{3.4}$$

where ρ_i are Hölder continuous positive densities on W_i. We refer to each (W_i, ρ_i) as a *standard pair*; this idea of working with pairs of admissible manifolds and densities was introduced by Chernov and Dolgopyat and is an important recent development in the study of SRB measures via geometric techniques.

To proceed in a more formal way, fix constants $\gamma, \kappa, r > 0$, and define a (γ,κ)-*admissible manifold of size* r to be $V(x) = \exp_x \operatorname{graph}\psi$, where

$\psi : B_{E^u(x)}(0, r) = B(0, r) \cap E^u(x) \to E^s(x)$ is $C^{1+\alpha}$ and satisfies

$$\begin{aligned} &\psi(0) = 0 \text{ and } D\psi(0) = 0, \\ &\|D\psi\| := \sup_{\|v\|<r} \|D\psi(v)\| \leq \gamma, \\ &|D\psi|_\alpha := \sup_{\|v_1\|,\|v_2\|<r} \frac{\|D\psi(v_1) - D\psi(v_2)\|}{\|v_1 - v_2\|^\alpha} \leq \kappa. \end{aligned} \tag{3.5}$$

Now fix $L > 0$ and write $\mathbf{K} = (\gamma, \kappa, r, L)$ for convenience. Then the space of admissible manifolds

$$\mathbb{R}_{\mathbf{K}} = \{\exp_x(\text{graph}\psi) \mid x \in U, \psi \in B_{E^u(x)}(r) \to E^s(x) \text{ satisfies (4.5)}\}$$

and the space of standard pairs

$$\mathbb{R}'_{\mathbf{K}} = \{(W, \rho) \mid W \in \mathbb{R}R_{\mathbf{K}}, \rho \in C^\alpha(W, [\tfrac{1}{L}, L]), |\rho|_\alpha \leq L\}$$

can be shown to be compact in the natural product topology.

A standard pair determines a measure $\Psi(W, \rho)$ on $\overline{U}$ in the obvious way:

$$\Psi(W, \rho)(E) := \int_{E\cap W} \rho \, dm_W. \tag{3.6}$$

Moreover, each measure η on $\mathbb{R}'_{\mathbf{K}}$ determines a measure $\Phi(\eta)$ on $\overline{U}$ by

$$\begin{aligned} \Phi(\eta)(E) &= \int_{\mathbb{R}'_{\mathbf{K}}} \Psi(W, \rho)(E) \, d\eta(W, \rho) \\ &= \int_{\mathbb{R}'_{\mathbf{K}}} \int_{E\cap W} \rho(x) \, dm_W(x) \, d\eta(W, \rho). \end{aligned} \tag{3.7}$$

Write $\mathcal{M}(\overline{U})$ and $\mathcal{M}(\mathbb{R}'_{\mathbf{K}})$ for the spaces of finite Borel measures on $\overline{U}$ and $\mathbb{R}'_{\mathbf{K}}$, respectively. It is not hard to show that $\Phi\colon \mathcal{M}(\mathbb{R}'_{\mathbf{K}}) \to \mathcal{M}(\overline{U})$ is continuous; in particular, $\mathcal{M}_{\mathbf{K}} = \Phi(\mathcal{M}_{\leq 1}(\mathbb{R}'_{\mathbf{K}}))$ is compact, where we write $\mathcal{M}_{\leq 1}$ for the space of measures with total weight at most 1.

On a uniformly hyperbolic attractor, an invariant probability measure is an SRB measure if and only if it is in $\mathcal{M}_{\mathbf{K}}$ for some $\mathbf{K}$. We see from (3.4) that $\mathcal{M}_{\mathbf{K}}$ is invariant under the action of f_*, and thus $\mu_n \in \mathcal{M}_{\mathbf{K}}$ for every n. By compactness of $\mathcal{M}_{\mathbf{K}}$, one can pass to a convergent subsequence $\mu_{n_k} \to \mu \in \mathcal{M}_{\mathbf{K}}$, and this is the desired SRB measure.

3.4.2. *Constructing SRB measures with effective hyperbolicity.* Now we move to the setting of our Main Theorems. One can show that the hypotheses of the second theorem imply the hypotheses of the first, so here we consider a d_u-dimensional manifold $W \subset U$ for which $m_W(S) > 0$, where we write S for the set of effectively hyperbolic points $x \in W$ with the property that $T_xW \subset K^u(x)$. In this setting, there are two major obstacles to overcome.

(1) The action of f along admissible manifolds is not necessarily uniformly expanding.

(2) Given $n \in \mathbb{N}$ it is no longer necessarily the case that $f^n(W)$ contains any admissible manifolds in $\mathbb{R}_{\mathbf{K}}$, let alone that it can be covered by them. When $f^n(W)$ contains some admissible manifolds, we will need to control how much of it can be covered.

To address the first of these obstacles, we need to consider admissible manifolds for which we control not only the geometry but also the dynamics; thus we will replace the collection $\mathbb{R}_{\mathbf{K}}$ from the previous section with a more carefully defined set (in particular, $\mathbf{K}$ will include more parameters). Since we do not have uniformly transverse invariant subspaces $E^{u,s}$, our definition of an admissible manifold also needs to specify which subspaces are used, and the geometric control requires an assumption about the angle between them.

Given $\theta, \gamma, \kappa, r > 0$, write $\mathbf{I} = (\theta, \gamma, \kappa, r)$ and consider the following set of (γ, κ)-*admissible manifolds of size* r *with transversals controlled by* θ:

$$
\begin{aligned}
(3.8)\quad \mathcal{P}_{\mathbf{I}} = \{\exp_x(\mathrm{graph}\psi) \mid x \in \overline{f(U)},\ T_xM = G \oplus F,\ G \subset \overline{K^u(x)},\\
\measuredangle(G, F) \geq \theta,\ \psi \in C^{1+\alpha}(B_G(r), F) \text{ satisfies (4.5)}\}.
\end{aligned}
$$

Elements of $\mathcal{P}_{\mathbf{I}}$ are admissible manifolds with controlled geometry. We also impose a condition on the dynamics of these manifolds. Fixing $C, \bar{\lambda} > 0$, write $\mathbf{J} = (C, \bar{\lambda})$ and consider for each $N \in \mathbb{N}$ the collection of sets

$$
\begin{aligned}
(3.9)\quad \mathcal{Q}_{\mathbf{J},N} = \{f^N(V_0) \mid V_0 \subset U, \text{ and for every } y, z \in V_0, \text{ we have}\\
d(f^j(y), f^j(z)) \leq Ce^{-\bar{\lambda}(N-j)} d(f^N(y), f^N(z)) \text{ for all } 0 \leq j \leq N\}.
\end{aligned}
$$

Elements of $\mathcal{P}_{\mathbf{I}} \cap \mathcal{Q}_{\mathbf{J},N}$ are admissible manifolds with controlled geometry and dynamics in the unstable direction. We also need a parameter $\beta > 0$ to control the dynamics in the stable direction, and a parameter $L > 0$ to control densities in standard pairs (as before). Then writing $\mathbf{K} = \mathbf{I} \cup \mathbf{J} \cup \{\beta, L\}$, we define in (Id**??**) a set $\mathbb{R}_{\mathbf{K},N} \subset \mathcal{P}_{\mathbf{I}} \cap \mathcal{Q}_{\mathbf{J},N}$ for which we have the added restriction to control the dynamics in the stable direction; the corresponding set of standard pairs will be written $\mathbb{R}'_{\mathbf{K},N}$.

The set $\mathbb{R}'_{\mathbf{K},N}$ carries a natural product topology; an element of $\mathbb{R}'_{\mathbf{K},N}$ is specified by a quintuple (x, G, F, ψ, ρ), and a small neighborhood $\Omega \supset x$ on can be identified with $\mathbb{R}^n$ via the exponential map. Then the second coordinate can be identified with the set of all k-dimensional subspaces of $\mathbb{R}^n$, the third with all $(n-k)$-dimensional subspaces, the fourth with C^1 functions $B_{\mathbb{R}^k}(r) \to \mathbb{R}^{n-k}$, and the fifth with C^0 functions $B_{\mathbb{R}^k}(r) \to [\frac{1}{L}, L]$. This specifies a natural topology on each coordinate: the Grassmanian topology on subspaces, and the C^1 and C^0 topologies on functions. Thus we may define a topology on $\mathbb{R}'_{\mathbf{K},N}$ as the product topology over each such Euclidean neighborhood in U. One can show that $\mathbb{R}'_{\mathbf{K},N}$ is compact in this topology and that the map Φ defined in (3.7) is continuous.

Let $\mathcal{M}_{\le 1}(\mathbb{R}'_{\mathbf{K},N})$ denote the space of measures on $\mathbb{R}'_{\mathbf{K},N}$ with total weight at most 1. The resulting measures on U will play a central role:

$$\mathcal{M}_{\mathbf{K},N} = \Phi(\mathcal{M}_{\le 1}(\mathbb{R}'_{\mathbf{K},N})). \tag{3.10}$$

One should think of $\mathcal{M}_{\mathbf{K},N} \subset \mathcal{M}(\overline{U})$ as an analogue of the regular level sets that appear in Pesin theory. Measures in $\mathcal{M}_{\mathbf{K},N}$ have uniformly controlled geometry, dynamics, and densities via the parameters in $\mathbf{K}$, and $\mathcal{M}_{\mathbf{K},N}$ is compact. However, at this point we encounter the second obstacle mentioned above: because $f(W)$ may not be covered by admissible manifolds in $\mathbb{R}_{\mathbf{K},N}$, the set $\mathcal{M}_{\mathbf{K},N}$ is not f_*-invariant.

Thus we must establish good recurrence properties to $\mathcal{M}_{\mathbf{K},N}$ under the action of f_* on $\mathcal{M}(\overline{U})$; this will be done via effective hyperbolicity. Consider for $x \in A$ and $\bar\lambda > 0$ the set of *effective hyperbolic times*.

$$\Gamma^e_{\bar\lambda}(x) = \left\{ n \mid \sum_{j=k}^{n-1} (\lambda^u - \Delta)(f^j(x)) \ge \bar\lambda(n-k) \text{ for all } 0 \le k < n \right\}. \tag{3.11}$$

One can show that for every x and almost every effective hyperbolic time $n \in \Gamma^e_{\bar\lambda}(x)$, there is a neighborhood $W^x_n \subset W$ containing x s.t. $f^n(W^x_n) \in \mathcal{P}_{\mathbf{I}} \cap \mathcal{Q}_{\mathbf{J},N}$. With a little more work, one can produce a **uniformly large** set of points x and times n s.t. $f^n(W^x_n) \in \mathbb{R}_{\mathbf{K},N}$, and in fact $f^n_* m_{W^x_n} \in \mathcal{M}_{\mathbf{K},N}$.

We use this to obtain measures $\nu_n \in \mathcal{M}_{\mathbf{K},N}$ s.t.

$$\nu_n \le \mu_n = \frac{1}{n} \sum_{k=0}^{n-1} f^k_* m_W \qquad \text{and} \qquad \limsup_{n\to\infty} \|\nu_n\| > 0. \tag{3.12}$$

Once this is achieved, we can use compactness of $\mathcal{M}_{\mathbf{K},N}$ to conclude that there is a non-trivial $\nu \in \bigcap_N \mathcal{M}_{\mathbf{K},N}$ s.t. $\nu \le \mu = \lim_k \mu_{n_k}$. In order to apply the absolute continuity properties of ν to the measure μ, we define a collection $\mathcal{M}^{ac}$ of measures with good absolute continuity properties along admissible manifolds, for which we can prove a version of the Lebesgue decomposition theorem that gives $\mu = \mu^{(1)} + \mu^{(2)}$, $0 \ne \nu \le \mu^{(1)}$, and the definition of $\mathbb{R}'_{\mathbf{K},N}$ will guarantee that the set of points with non-zero Lyapunov exponents has positive measure with respect to ν, and hence also with respect to $\mu^{(1)}$. Thus some ergodic component of $\mu^{(1)}$ is hyperbolic, and hence is an SRB measure.

4. SRB measures for hyperbolic attractors with singularities

4.1. Topological attractors with singularities. M smooth compact manifold, $U \subset M$ an open bounded connected subset, *the trapping region*, $N \subset U$ a closed subset and $f : U \setminus N \to U$ a C^2 diffeomorphism s.t.

$$\begin{aligned} \|d^2 f_x\| &\le C_1 d(x, \mathcal{S}^+)^{-\alpha_1} \text{ for any } x \in U \setminus N, \\ \|d^2 f_x^{-1}\| &\le C_2 d(x, \mathcal{S}^-)^{-\alpha_2} \text{ for any } x \in f(U \setminus N), \end{aligned} \tag{4.1}$$

where $\mathcal{S}^+ = N \cup \partial U$ is the *singularity set for* f and $\mathcal{S}^- = f(\mathcal{S}^+)$ that is

$\mathcal{S}^- = \{y \in U :$ there is $z \in \mathcal{S}^+$ and $z_n \in U \backslash \mathcal{S}^+$ such that $z_n \to z, f(z_n) \to f(z)\}$ is the *singularity set for* f^{-1}. We will assume that $m(\mathcal{S}^+) = m(\mathcal{S}^-) = 0$.

Define

$$U^+ = \{x \in U : f^n(x) \notin \mathcal{S}^+, n = 1, 2, \dots\}$$

and the *topological attractor with singularities*

$$D = \bigcap_{n \ge 0} f^n(U^+), \quad \Lambda = \bar{D}.$$

Given $\varepsilon > 0$ and $\ell > 1$, set

$$\begin{aligned} D^+_{\varepsilon,\ell} &= \{z \in \Lambda : d(f^n(z), \mathcal{S}^+) \ge \ell^{-1} e^{-\varepsilon n}, n = 0, 1, 2, \dots\}, \\ D^-_{\varepsilon,\ell} &= \{z \in \Lambda : d(f^n(z), N^-) \ge \ell^{-1} e^{-\varepsilon n}, n = 0, 1, 2, \dots\}, \\ D^0_{\varepsilon,\ell} &= D^+_{\varepsilon,\ell} \bigcap D^-_{\varepsilon,\ell}, \\ D^0_\varepsilon &= \bigcup_{\ell \ge 1} D^0_{\varepsilon,\ell}. \end{aligned}$$

The set D^0_ε is the *core* of the attractor and it may be an empty set as it may be the set D.

Theorem 4.1. *Assume that there are $C > 0$ and $q > 0$ s.t. for any $\varepsilon > 0$ and $n > 0$*

$$m(f^{-n}(\mathcal{U}(\varepsilon, \mathcal{S}^+) \cap f^n(U^+))) \le C\varepsilon^q, \tag{4.2}$$

where $\mathcal{U}(\varepsilon, \mathcal{S}^+)$ is a neighborhood of the (closed) set $\mathcal{S}^+$. Then there is an invariant measure μ on Λ s.t., $\mu(D^0_\varepsilon) > 0$, in particular, the core is not empty.

Proof. Let $\nu_n = f^n_* m$. Since $m(\mathcal{S}^+) = 0$, this measure is well defined in $\bar{U}$ and so is the sequence of measures

$$\mu_n = \frac{1}{n} \sum_{k=0}^{n-1} \nu_k,$$

which is compact in the week*-topology. Let μ be a limit measure. Consider the set

$$\hat{D}^+_{\varepsilon,\ell} = \{z \in \Lambda : d(f^n(z), \mathcal{S}^+) \ge \ell^{-1} e^{-\varepsilon n},\ n = 0, 1, 2, \dots\}.$$

Choose $\gamma > 0$ and $\varepsilon > 0$. For $\ell > \gamma^{-1}$

$$U \setminus \hat{D}^+_{\varepsilon,\ell} \subset \{x \in U : \text{ there exists } i > 0 \text{ s.t. } f^i(x) \in \mathcal{U}(\gamma e^{-\varepsilon i}, \mathcal{S}^+)\}.$$

Therefore, Assumption (4.2) implies

$$\begin{aligned}\nu_n(U \setminus \hat{D}^+_{\varepsilon,\ell}) &\le \sum_{i=0}^{\infty} m(f^{-i}(\mathcal{U}(\gamma e^{-\varepsilon i}, \mathcal{S}^+) \cap f^i(U))) \\ &\le C\gamma^q \sum_{i=0}^{\infty} e^{-q\varepsilon i} \le C_1\gamma^q,\end{aligned}$$

where $C_1 > 0$ is a constant. It follows that

$$\mu_n(\hat{D}^+_{\varepsilon,\ell}) \ge 1 - C_1\gamma^q$$

for all large enough ℓ. Since the sets $\hat{D}^+_{\varepsilon,\ell}$ are closed we have that

$$\mu(D^+_{\varepsilon,\ell}) = \mu(\hat{D}^+_{\varepsilon,\ell}) \ge 1 - C_1\gamma^q.$$

We conclude that $\mu(D^+_\varepsilon) = 1$ and hence, $\mu(\mathcal{S}^+) = 1$ and the desired result follows.

4.2. **Hyperbolic attractors with singularities.** We say that a topological attractor with singularities Λ is *hyperbolic*, if there exist two families of stable and unstable cones

$$K^s(x) = K(x, E_1(x), \theta(x)),\ K^u(x) = K(x, E_2(x), \theta(x)),\ x \in U \setminus \mathcal{S}^+$$

s.t.

(1) the angle $\angle(E_1(x), E_2(x)) \ge$ const. ;
(2) $df(K^s(x)) \subset K^s(f(x))$ for any $x \in U \setminus \mathcal{S}^+$ and $df^{-1}(K^u(x)) \subset K^u(f(x))$ for any $x \in f(U \setminus \mathcal{S}^+)$;
(3) for some $\lambda > 1$
   (a) $\|df_x v\| \ge \lambda \|v\|$ for $x \in U \setminus \mathcal{S}^+$ and $v \in K^u(x)$;
   (b) $\|df_x^{-1} v\| \ge \lambda \|v\|$ for $x \in f(U \setminus \mathcal{S}^+)$ and $v \in K^s(x)$.

Theorem 4.2. *Let Λ be a hyperbolic attractor with singularities for a $C^{1+\alpha}$ map and assume that Condition* (4.2) *holds. Then f admits an SRB measure on Λ.*

4.3. **Examples.** We describe the following three examples of hyperbolic attractors with singularities which satisfy requirement (4.2).

The Lorenz attractor. Let $I = (-1, 1)$, $U = I \times I$, $N = I \times 0 \subset U$ and $f : U \setminus N \to U$ is given by

$$f(x, y) = ((-B|y|^{\nu_0} + B s\,\mathrm{sign}(y)|y|^{\nu} + 1)\mathrm{sign}(y),\ ((1 + A)|y|^{\nu_0} - A)\mathrm{sign}(y)),$$

where

$$0 < A < 1, \quad 0 < B < \frac{1}{2}, \quad \nu > 1, \quad \frac{1}{1 + A} < \nu_0 < 1.$$

This attractor appears in the Lorenz system of ODE :

$$\dot{x} = -\sigma x + \sigma y, \quad \dot{y} = rx - y - xz, \quad \dot{z} = xy - bz$$

for the values of the parameters $\sigma = 10$, $b = \frac{8}{3}$ and $r \sim 24.05$

The Lozi attractor. Let $I = (-c, c)$ for some $0 < c < 1$ and let $U = I \times I$, $N = 0 \times I \subset U$ and $f : U \setminus N \to U$ is given by

$$f(x, y) = (1 + by - a|x|, x),$$

where $0 < a < a_0$ and $0 < b < b_0$ for some small $a_0 > 0$ and $b_0 > 0$.

Up to a change of coordinates this map was introduced by Lozi as a simple version of the famous Hénon map in population dynamics.

The Belykh attractor. Let $I = (-1, 1)$, $U = I \times I$, $N = \{(x, y) : y = kx\} \subset U$ and $f : U \setminus N \to U$ is given by

$$f(x, y) = \begin{cases} (\lambda_1(x - 1) + 1,\ \lambda_2(y - 1) + 1) & \text{for } y > kx, \\ (\mu_1(x + 1) - 1,\ \mu_2(y + 1) - 1) & \text{for } y < kx, \end{cases}$$

where

$$0 < \lambda_1, \mu_1 < \frac{1}{2}, \quad 1 < \lambda_2, \mu_2 < \frac{2}{1 - |k|}, \quad |k| < 1.$$

If $\lambda_1 = mu_1$ and $\lambda_2 = \mu_2$ this map was introduced by Belykh as one of the simplest models in the phase synchronization theory in radiophysics.

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