Mathematical Statistical Physics – LMU München, summer semester 2012

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Homework sheet 4 – due on 18.05.2012 – and exercises for the class on 11.05.2012

4.1 Γ *function and polar coordinates practice.* Calculate the ($(n-1)$-dimensional) surface volume $s_n(r)$ of the n-dimensional sphere with radius r, for every positive integer n in terms of the Γ function defined as

$$\Gamma(x) := \int_0^\infty t^{x-1} e^{-t}\,\mathrm{d}t.$$

hint: integrate $f(x_1,\dots,x_n) = \frac{1}{\sqrt{2\pi}^n} e^{-\frac{x_1^2+\dots+x_n^2}{2}}$ *on* $\mathbb{R}^n$.

4.2 Let the random vector $(v_1, v_2, \dots, v_N)$ be uniformly distributed on the (surface of the) N-dimensional sphere with radius $\sqrt{2NE}$, where $E \in (0,\infty)$ is a fixed number. Find the limit distribution of v_1 as $N \to \infty$. *hint: calculate the density for each* N *using the result of Exercise 1, then use the Stirling formula*

$$\Gamma(x) = \sqrt{\frac{2\pi}{x}} \left(\frac{x}{e}\right)^x (1 + o(1)).$$

4.3 **(homework)** Let the random vector $(v_1, v_2, \dots, v_N)$ be uniformly distributed on the simplex

$$\{(v_1,\dots,v_N) \in \mathbb{R}^N \,:\, 0 \le v_i,\, v_1 + \dots + v_N = NE\},$$

where $E \in (0,\infty)$ is a fixed number. Find the limit distribution of v_1 as $N \to \infty$.

4.4 We roll a fair die 10 times and record the results. Let X be the random 10-digit number we get. Calculate the entropy of X.

4.5 **(homework)** We toss a biased coin with $\mathbb{P}(heads) = p \in (0,1)$ 10 times and record the results. Let Y be the random 10-long string we get. Calculate the entropy of Y.

4.6 *Maximum entropy principle.* The maximum entropy principle describes the probability measures that have maximum relative entropy w.r.t some reference measure under certain constraints – namely, with the integrals of certain (arbitrary) functions being pre-given:

Theorem 1 (Maximum entropy principle) *Let* $(\Omega, \mathcal{F}, \nu)$ *be a (not necessarily probability) measure space. Suppose that* $X_1,\dots,X_n$ *are pre-given measurable (real-valued) functions on* $(\Omega,\mathcal{F})$ *and* $m_1,\dots,m_n$ *are pre-given real numbers. We consider those probability measures on* $(\Omega,\mathcal{F})$*, w.r.t. which the integrals of our pre-given functions are exactly the pre-given numbers:*

$$\mathcal{P}(\underline{X},\underline{m}) := \left\{ \mu \text{ probability measure on } (\Omega,\mathcal{F}) \,:\, \int_\Omega X_i \,\mathrm{d}\mu = m_i \text{ for } i = 1,\dots,n \right\}.$$

Suppose that we can choose $t_1,\dots,t_n \in \mathbb{R}$ *with the following properties:*

- $Z_{\underline{t}} := \int_\Omega e^{-\sum_{i=1}^n t_i X_i(\omega)}\,\mathrm{d}\nu(\omega) < \infty$,
- *the probability measure* $\mu_{\underline{t}}$ *on* $(\Omega,\mathcal{F})$ *which is absolutely continuous w.r.t.* ν*, with density* $\rho_{\underline{t}}(\omega) := \frac{1}{Z_{\underline{t}}} e^{-\sum_{i=1}^n t_i X_i(\omega)}$ *satisfies* $\mu_{\underline{t}} \in \mathcal{P}(\underline{X},\underline{m})$*. Then* $\mu_{\underline{t}}$ *is the (unique) probability measure in* $\mathcal{P}(\underline{X},\underline{m})$ *which has maximal entropy w.r.t* ν*, and*

$$S(\mu_{\underline{t}};\nu) = \sum_{i=1}^n t_i m_i + \log Z_{\underline{t}}.$$

Use this theorem to find (the distribution of) the random variable X with maximum entropy (if it exist)

(a) w.r.t. Lebesgue measure on $\mathbb{R}$, under the constraint $\mathbb{E}X = m$,

(b) w.r.t. Lebesgue measure on $\mathbb{R}^+$, under the constraint $\mathbb{E}X = m$,

(c) **(homework)** w.r.t. Lebesgue measure on $\mathbb{R}$, under the constraints $\mathbb{E}X = m$, $\text{Var}X = v$,

(d) **(homework)** w.r.t. the counting measure on $\mathbb{N}$, under the constraint $\mathbb{E}X = m$.

4.7 **(homework)** *Microcanonical description of the free gas.* Consider N identical particles of mass m in a box $\Lambda \subset \mathbb{R}^3$ (with volume V), with the Hamiltonian

$$H(\underline{q}, \underline{p}) = \sum_{i=1}^{N} \frac{\vec{p_i}^2}{2m}$$

(the particles are non-interacting). Fix the total energy to be E.

(a) Describe the microcanonical distribution $\mu_{\text{micr}} = \mu_{N,V,E}$.

(b) Calculate the microcanonical partition function $Z_{\text{micr}} = Z(N, V, E)$. (Use the result of Exercise 1.)

(c) Calculate the entropy $S(N, V, E)$ of μ_{micr} (relative to the "natural reference measure", which is the conditional measure of the Lebesgue measure (of the phase space) on the $\{H = E\}$ surface).

(d) Set $E = Nu$, $V = Nv$ with u, v fixed constants, so $S(N, V, E)$ becomes $S_{u,v}(N)$. How does $S_{u,v}(N)$ scale with N? Use the Stirling formulas

$$\Gamma(x) = \sqrt{\frac{2\pi}{x}} \left(\frac{x}{e}\right)^x (1 + o(1)) \quad , \quad n! = \sqrt{2\pi n} \left(\frac{n}{e}\right)^n (1 + o(1)).$$

Have you not forgotten to factorize the phase space due to the indistinguishability of the particles?